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JHEP07(2009)003

(http://iopscience.iop.org/1126-6708/2009/07/003)

Published by IOP Publishing for SISSA

Received: *March 9, 2009*
Accepted: *June 7, 2009*
Published: *July 1, 2009*

Universal BPS structure of stationary supergravity solutions

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Abstract: We study asymptotically flat stationary solutions of four-dimensional supergravity theories via the associated $\mathfrak{G}/\mathfrak{H}^*$ pseudo-Riemannian non-linear sigma models in three spatial dimensions. The Noether charge $\mathscr{C}$ associated to $\mathfrak{G}$ is shown to satisfy a characteristic equation that determines it as a function of the four-dimensional conserved charges. The matrix $\mathscr{C}$ is nilpotent for non-rotating extremal solutions. The nilpotency degree of $\mathscr{C}$ is directly related to the BPS degree of the corresponding solution when they are BPS. Equivalently, the charges can be described in terms of a Weyl spinor $|\mathscr{C}\rangle$ of $\mathrm{Spin}^*(2\mathcal{N})$, and then the characteristic equation becomes equivalent to a generalisation of the Cartan pure spinor constraint on $|\mathscr{C}\rangle$. The invariance of a given solution with respect to supersymmetry is determined by an algebraic 'Dirac equation' on the Weyl spinor $|\mathscr{C}\rangle$. We explicitly solve this equation for all pure supergravity theories and we characterise the stratified structure of the moduli space of asymptotically Taub–NUT black holes with respect to their BPS degree. The analysis is valid for any asymptotically flat stationary solutions for which the singularities are protected by horizons. The $\mathfrak{H}^*$-orbits of extremal solutions are identified as Lagrangian submanifolds of nilpotent orbits of $\mathfrak{G}$, and so the moduli space of extremal spherically symmetric black holes is identified as a Lagrangian subvariety of the variety of nilpotent elements of $\mathfrak{g}$. We also generalise the notion of active duality transformations to an 'almost action' of the three-dimensional duality group $\mathfrak{G}$ on asymptotically flat stationary solutions.

Keywords: Black Holes, Supergravity Models, Global Symmetries

ArXiv ePrint: 0902.4438

doi:10.1088/1126-6708/2009/07/003

Contents

1 Introduction

Black hole solutions of supergravity theories have been extensively studied in the literature [1, 2]. This applies in particular to BPS solutions, that is, to supersymmetric solutions admitting Killing spinors (see e.g. [3] and references therein). In this publication, we focus on theories for which the scalar fields lie in a symmetric space, which can be represented as the quotient of the global hidden symmetry by a maximal subgroup [1, 4, 5]. Stationary solutions can then be identified as solutions of a three-dimensional non-linear sigma model over a symmetric space $\mathfrak{G}/\mathfrak{H}^*$ coupled to Euclidean gravity, such that the maximal subgroup $\mathfrak{H}^*$ is *non-compact*. The group $\mathfrak{G}$ extends the global 'hidden symmetry' group $\mathfrak{G}_4$ of four-dimensional supergravity; while the latter acts only on matter degrees of freedom, the larger group $\mathfrak{G} \supset \mathfrak{G}_4$ also incorporates gravity and the Ehlers $SL(2, \mathbb{R})$ symmetry [6].

For a large class of theories, the duality group $\mathfrak{G}$ is simple, and then all the non-extremal black hole solutions are in $\mathfrak{H}^*$-orbits of solutions of pure Einstein theory; indeed, all of these solutions can be obtained through group theoretical methods [1, 2]. The Noether current associated to the duality symmetry gives rise to a charge $\mathscr{C}$ lying in the Lie algebra $\mathfrak{g} \equiv \mathrm{Lie}(\mathfrak{G})$. We display the physical content of this charge in terms of four-dimensional quantities. Imposing regularity conditions on the solution requires that $\mathscr{C}$ satisfy a characteristic equation which determines $\mathscr{C}$ non-linearly as a function of the conserved charges of the four-dimensional theory, i.e. the Komar mass, the NUT charge and the electromagnetic charges. Consequently, they transform all together in a non-linear representation of the group $\mathfrak{H}^*$.

Our results are based on an extension of the general classification of three-dimensional supergravity theories [7] to theories with Euclidean signature which characterise the stationary solutions of supergravity theories in four dimensions. For $\mathcal{N}$-extended supergravity, the group $\mathfrak{H}^*$ is the product of the group $\mathrm{Spin}^*(2\mathcal{N})$ (for $\mathcal{N} > 1$, a non-compact real form of the group $\mathrm{Spin}(2\mathcal{N})$ appearing for Lorentzian supergravities [7]) with a symmetry group determined by the matter content of the theory. The charge matrix $\mathscr{C}$ can then be associated to a charge state vector $|\mathscr{C}\rangle$ which transforms as a $\mathrm{Spin}^*(2\mathcal{N})$ chiral spinor. For asymptotically flat solutions (flat in the sense of Misner [8], i.e. including the asymptotically Taub–NUT ones) the BPS condition is equivalent to an algebraic 'Dirac equation' (see (2.45) for a precise formulation)

$$\not{\epsilon}\,|\mathscr{C}\rangle = 0 \tag{1.1}$$

where the 'momentum' ϵ is the asymptotic supersymmetry parameter (Killing spinor) transforming in the (pseudo-real) vector representation of $SO^*(2\mathcal{N})$. This equation follows from the dilatino variation, which in three dimensions carries all the essential information about residual supersymmetries. Equally important, (1.1) determines the charge $\mathscr{C}$ for BPS solutions as a function of the conserved charges of the four-dimensional theory in terms of a simple rational function, whereas the generic non-BPS solution of the characteristic equation is generally a non-rational function. For $\mathcal{N} \leq 5$ pure supergravities, the characteristic equation for the charge matrix $\mathscr{C}$ simply reduces to the $\mathrm{Spin}^*(2\mathcal{N})$ pure spinor equation for the charge spinor $|\mathscr{C}\rangle$, and can be solved in full generality.

In order to identify the general conditions on charges for various BPS degrees, we solve equation (1.1) systematically for all pure supergravity theories (the $\mathcal{N} = 4$ theory coupled to n vector multiplets is also considered in the last section). In this way we are able to provide a systematic classification of BPS solutions for all supergravities. Our analysis encompasses previous work on BPS solutions, such as, for instance, the BPS asymptotically Minkowskian stationary black holes solutions in $\mathcal{N} = 2$ supergravity [9, 10], whose classification became possible through the discovery of the so-called attractor mechanism [11] (see [12] for an introduction to the subject, and [13] for an extension of these results including R^2 corrections). In addition (and amongst other results) we are able to prove the conjecture of [14] on the vanishing of the horizon area for $\frac{1}{4}$ and $\frac{1}{2}$ BPS solutions to $\mathcal{N} = 8$ supergravity. We then conjecture an expression for the horizon area of asymptotically Taub–NUT BPS black holes, which turns out not to be $E_{7(7)}$ invariant in general. Moreover, neither the horizon area, nor the surface gravity are invariant under the action of $\mathrm{Spin}^*(16)$, but the product of these two is proportional to the square root of the $E_{8(8)}$ invariant $\mathrm{Tr}\, \mathscr{C}^2$.[1]

The moduli space of stationary single-particle solutions admits a stratified structure whose filtration corresponds to the BPS degree in pure supergravity theories with $\mathcal{N} \leq 5$. The strata of BPS degrees $(n/\mathcal{N})$ then can be given as coset spaces $\mathfrak{H}^*/\mathfrak{I}_n$ which we identify explicitly in terms of the isotropy subgroups $\mathfrak{I}_n \subset \mathfrak{H}^*$ of the given charges. We also describe the moduli space of stationary single-particle solutions of $\mathcal{N} = 6$ and $\mathcal{N} = 8$ supergravities. In these cases, the stratification is slightly more involved. We show that the BPS degree is characterised in a $\mathfrak{G}$-invariant way by the nilpotency degree of the charge matrix $\mathscr{C}$ in both the fundamental and the adjoint representation of $\mathfrak{g}$. Another main new result of this work is the demonstration that these BPS strata, initially obtained as orbits of the asymptotic-structure-preserving group $\mathfrak{H}^*$, are diffeomorphic to *Lagrangian submanifolds of nilpotent orbits under the adjoint action of* $\mathfrak{G}$.

We also generalise the notion of active duality transformations [15] to the case of three-dimensional Euclidean theories. Unlike in four dimensions, this procedure fails to define a Lie group action because of the singular behaviour of the action on the BPS solutions. This failure is directly related to the failure of the Iwasawa decomposition when the maximal subgroup $\mathfrak{H}^* \subset \mathfrak{G}$ is non-compact: the elements of $\mathfrak{G}$ mapping non-BPS to BPS solutions are precisely the ones for which the Iwasawa decomposition breaks down. We will explain in some detail how the BPS strata are related to the 'Iwasawa failure sets' in $\mathfrak{G}$.

A chief motivation for the present work was provided by the general conjecture of [16] (see also [17]) according to which the global hidden symmetries $\mathfrak{G}$ of supergravity become replaced by certain arithmetic subgroups $\mathfrak{G}(\mathbb{Z}) \subset \mathfrak{G}$ in the quantum theory,[2] and to explore whether and in what sense this claim can remain valid as one descends to three dimensions. This case cannot be simply extrapolated from higher dimensional examples, because it differs from those in two crucial respects: (1) unlike in dimensions $d \geq 4$, the central charges of the superalgebra no longer combine into representations of the global hidden symmetry group $\mathfrak{G}$ [18], and (2) the quantisation condition would now also apply to the gravitational charges (mass and NUT charge) [19]. For maximal supergravity, our analysis leads us to the conclusion that the quantum moduli space of maximal supergravity solitons is *not* described by a lattice in the adjoint representation of an arithmetic subgroup of $E_{8(8)}$.

[1] For rotating solutions this also involves the angular momentum per unit of mass, which is also left invariant by the action of $\mathrm{Spin}^*(16)$.

[2] The main example here is maximal $\mathcal{N} = 8$ supergravity [4] whose global symmetry $\mathfrak{G}_4 = E_{7(7)}$ is broken to $E_7(\mathbb{Z}) = E_{7(7)} \cap \mathrm{Sp}(56, \mathbb{Z})$ upon quantisation, where the symplectic group $\mathrm{Sp}(56, \mathbb{Z})$ encodes the Dirac–Schwinger–Zwanziger quantisation condition for the electromagnetic charges.

We will argue that the singular behaviour of the duality transformations on the subset of BPS solutions within the space of all stationary solutions might be resolved at the quantum level. This conjecture is based on our description of the $\mathfrak{H}^*$-orbits as Lagrangian submanifolds of the $\mathfrak{G}$-adjoint orbits of the corresponding charge matrix: if there is no representation of the duality group $\mathfrak{G}$ on the moduli space of asymptotically flat stationary solutions, there might nevertheless exist a unitary representation of $\mathfrak{G}$ on the space of functions defined on this moduli space. The action of $\mathfrak{G}$ on the adjoint orbits induces a unitary representation on the spaces of functions supported on these Lagrangian submanifolds, that is on the moduli space of solutions. We speculate on an interpretation of the formula for the Eisenstein series obtained from the minimal unitary representation of $\mathfrak{G}$ [20, 21] in terms of observables of the quantum mechanics of a particle living in the moduli space of $\frac{1}{2}$ BPS black holes. Whereas such a construction of the Eisenstein series seems meaningful in the study of the moduli space of particle solutions, the naïve generalisation of higher dimensional formulas for Eisenstein series as sums over a lattice representation of $\mathfrak{G}(\mathbb{Z})$ should not be interpreted as a sum over the quantised charges.

An interesting problem for future study will be the extension of the present results to solutions with a lightlike Killing vector [22], which are plane-wave when they are BPS (see e.g. [23] and references therein). Here we only remark that, in four spacetime dimensions, *pp*-type plane wave solutions cannot be asymptotically flat because the solutions of the transverse Laplace equation decay only logarithmically.

2 Duality groups of stationary solutions

In Einstein theory coupled to matter, one generally knows exact solutions only when the corresponding metric admits a certain number of commuting Killing vectors, and when these isometries furthermore leave invariant the various gauge fields and matter fields of the theory. The existence of k such Killing vectors permits elimination of the dependance of the solution on k corresponding coordinates, in such a way that the solution can be interpreted in $(d–k)$ dimensions. Moreover, specific dimensionally reduced theories admit enlarged sets of symmetries which are non-linearly realised on the solutions. When all of the reduction Killing vectors are spacelike, the fields of a dimensionally reduced theory are defined on a $(d-k)$-dimensional spacetime and the Hamiltonian of the theory is positively defined. By contrast, when one of the Killing vectors is timelike, as a general property of timelike dimensional reductions, the action of the dimensionally reduced theory is indefinite [1, 2].[3] This is not a problem since we are here concerned only with the classical equations of motion in an Euclidean-signature reduced theory for stationary solutions.

We will consider Einstein theory coupled to abelian vector fields and scalar fields living in a symmetric space. We assume that the isometry group of the scalar coset space is a semi-simple Lie group $\mathfrak{G}_4$ which defines a symmetry of the equations of motion, and moreover that each simple or abelian group arising in the decomposition of $\mathfrak{G}_4$ acts non-trivially on the vector fields. The scalar coset space $\mathfrak{G}_4/\mathfrak{H}_4$ is a Riemannian manifold defined by the quotient of $\mathfrak{G}_4$ by its maximal compact subgroup $\mathfrak{H}_4$. The various Lie groups $\mathfrak{G}_4$

[3]Timelike dimensional reduction has also been used to generate solutions in [24].

satisfying these criteria are listed in [1]. We denote by $\mathfrak{l}_4$ the representation carried by the Maxwell degrees of freedom under $\mathfrak{G}_4$. If we consider only stationary solutions, we can consider them as solutions of a dimensionally reduced Euclidean three-dimensional theory. This dimensional reduction yields one scalar from the metric, one scalar for each Maxwell field, together with all the original scalars of the four-dimensional theory together with one vector field coming from the metric and one vector field from each Maxwell field. For a timelike Killing vector, the kinetic terms of tensor fields whose rank has been reduced by an odd number come with a negative sign, while the remaining fields' kinetic terms are positive (for a spacelike Killing vector, they would all appear with a positive sign). This holds, in particular, for the scalars arising from Maxwell fields ($1 \to 0$) and for the vector field arising from the metric ($2 \to 1$). After dualisation of the vector fields to scalars, the vector field arising from the metric turns into a scalar field with a *positive* kinetic term. The Maxwell vectors become scalars after dualisation, with negative kinetic terms similarly to the Maxwell scalars. We will call the latter 'electric' fields, and we will call the scalars arising from vectors upon dualisation 'magnetic' fields.

The stationary solutions of pure gravity admit the so-called Ehlers group [6] as a symmetry, yielding a formulation of the theory as an $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ non-linear sigma model coupled to three-dimensional gravity. This property generalises to Einstein theory with matter in such a way that we get an $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{g}_4$ set of symmetry generators. The scalars arising from the Maxwell fields admit a shift symmetry, since the Maxwell fields can be shifted by constants (global gauge transformations) already in four dimensions, so the $\mathfrak{G}_4$ symmetry is enhanced to the non-semi-simple group $\mathfrak{G}_4 \ltimes \mathfrak{l}_4$. After dimensional reduction to three dimensions, the magnetic scalars obtained by dualisation from the vectors also admit shift symmetries, since each dualisation leaves an undetermined integration constant. Altogether, these shift symmetries still transform in the $\mathfrak{l}_4$ of $\mathfrak{G}_4$. Moreover, the commutators of the Ehlers generators with these shift symmetries generate new generators which also belong to the $\mathfrak{l}_4$ representation of $\mathfrak{G}_4$, and which are themselves nonlinearly realised on the fields. The whole duality group then becomes a simple Lie group [1], for which the algebra admits a five-graded decomposition with respect to the diagonal generator of the Ehlers $\mathrm{SL}(2,\mathbb{R})$

$$\mathfrak{g} \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{g}_4 \oplus (\mathbf{2} \otimes \mathfrak{l}_4) \cong \mathbf{1}^{(-2)} \oplus \mathfrak{l}_4^{(-1)} \oplus (\mathbf{1} \oplus \mathfrak{g}_4)^{(0)} \oplus \mathfrak{l}_4^{(1)} \oplus \mathbf{1}^{(2)} \tag{2.1}$$

The maximal compact subgroup $\mathfrak{H}$ of this group is generated by the $\mathfrak{so}(2) \oplus \mathfrak{h}_4$ subalgebra of $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{g}_4$, together with the compact combination of the two $\mathfrak{l}_4$. Because the scalar fields arising from the Maxwell fields have negative kinetic terms, however, the maximal subgroup $\mathfrak{H}^*$ for a timelike dimensional reduction is a *non-compact real form* of this maximal compact subgroup [1], in contrast to spacelike reductions, for which $\mathfrak{H}$ is fully compact.

The resulting three-dimensional theory is described in terms of a coset representative $\mathcal{V} \in \mathfrak{G}/\mathfrak{H}^*$ which contains all the propagating (scalar) degrees of freedom of the theory, plus the three-dimensional metric[4] $\gamma_{\mu\nu}$ which, however, carries no physical degrees of freedom

[4] We denote curved spacetime indices by Greek letters $\mu, \nu, \ldots$ in both four and three dimensions.

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in three dimensions. The Maurer–Cartan form $\mathcal{V}^{-1}d\mathcal{V}$ decomposes as

$$\mathcal{V}^{-1}d\mathcal{V} = Q + P \quad , \qquad Q \equiv Q_\mu dx^\mu \in \mathfrak{h}^* \quad , \quad P \equiv P_\mu dx^\mu \in \mathfrak{g} \ominus \mathfrak{h}^* \tag{2.2}$$

The Bianchi identity then gives

$$dQ + Q^2 = -P^2 \, , \qquad d_Q P \equiv dP + \{Q, P\} = 0 \, . \tag{2.3}$$

The equations of motion of the scalar fields are

$$d_Q \star P \equiv d \star P + \{Q, \star P\} = 0 \tag{2.4}$$

and the Einstein equations are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \mathrm{Tr}\ P_\mu P_\nu \tag{2.5}$$

We will consider in this paper solutions which are asymptotically flat in the sense of Misner [8] (which are not generally to be confused with asymptotically Minkowskian solutions). Strictly speaking, by this we mean that four-dimensional spacetime admits a function r that tends to infinity at spatial infinity and which defines a proper distance in this limit, $g^{\mu\nu}\partial_\mu r \partial_\nu r \to 1$, and such that all the components of the Riemann tensor in any vierbein frame tend to zero as $\mathcal{O}(r^{-3})$ as $r \to +\infty$. In the same way, the Maxwell field strengths are required to tend to zero as $\mathcal{O}(r^{-2})$ in this limit in any vierbein frame.[5] The four-dimensional coset elements $\in \mathfrak{G}_4/\mathfrak{H}_4$ are required to tend asymptotically to the unit matrix as $\mathbb{1} + \mathcal{O}(r^{-1})$. In order for charges to be well defined, we also require that the timelike Killing vector $\kappa \equiv \kappa^\mu \partial_\mu$ leaves invariant the function r, that it satisfies asymptotic hypersurface orthogonality $\varepsilon^{\mu\nu\sigma\rho}\kappa_\nu \partial_\sigma \kappa_\rho = \mathcal{O}(r^{-2})$, and that its squared norm $-H$ tends to negative unity as $H = 1 + \mathcal{O}(r^{-1})$ in the asymptotic region. We assume that the action of the timelike isometry on the domain M_+ of the four-dimensional manifold M on which H is positively defined (i.e. outside possible horizons and ergospheres) is free and proper. M_+ then admits an abelian principal bundle structure whose fibres are the timelike isometry orbits and whose base is a three-dimensional Riemannian manifold V which is asymptotically Euclidean. For such specific solutions, all the fields of the four-dimensional theory can be defined from pull-backs of the scalar fields living in the symmetric space $\mathfrak{G}/\mathfrak{H}^*$ defined throughout V, and the asymptotic condition in the four-dimensional theory is equivalent to the requirement that the coset representative $\mathcal{V}$ goes to the unit matrix as $\mathcal{V} = \mathbb{1} + \mathcal{O}(r^{-1})$ in the asymptotic region.

2.1 Conserved charges

We define the Komar two-form $K \equiv \partial_\mu \kappa_\nu dx^\mu \wedge dx^\nu$ [25], which is invariant under the action of the timelike isometry and which is asymptotically horizontal. The latter condition is equivalent to the requirement that the scalar field B, dual to the Kaluza–Klein vector arising from the metric by dimensional reduction, vanishes as $\mathcal{O}(r^{-1})$ as $r \to +\infty$. We

[5] More generally, one could consider solutions tending asymptotically to an arbitrary $\mathfrak{G}_4/\mathfrak{H}_4$ constant matrix, but this can be standardised to the unit matrix by making a $\mathfrak{G}_4$ transformation.

define a set of local sections of the principal bundle on each open set of an atlas of the three-dimensional manifold V, which we denote collectively s. Then we can define the Komar mass and the Komar NUT charge as follows [26]

$$m \equiv \frac{1}{8\pi}\int_{\partial V} s^* \star K \qquad n \equiv \frac{1}{8\pi}\int_{\partial V} s^* K \tag{2.6}$$

The Maxwell fields also define conserved charges. The Maxwell equation $d\star\mathcal{F}=0$, where $\mathcal{F}\equiv\delta\mathcal{L}/\delta F$ is a linear combination of the two-form field strengths F depending on the four-dimensional scalar fields, permits one to define electric charges, and the Bianchi identities $dF=0$ permits one to define magnetic charges, as follows:

$$q \equiv \frac{1}{2\pi}\int_{\partial V} s^* \star \mathcal{F} \qquad p \equiv \frac{1}{2\pi}\int_{\partial V} s^* F \ . \tag{2.7}$$

These transform together in the representation $\mathfrak{l}_4$ of $\mathfrak{G}_4$. Finally, the rigid $\mathfrak{G}_4$ invariance of the four-dimensional theory gives rise to an associated conserved current such that the associated three-form J_3 transforms in the adjoint representation of $\mathfrak{G}_4$, and satisfies $dJ_3=0$ if the scalar field equations are obeyed. However, J_3 cannot generally be written as a local function of the fields and their derivatives in four dimensions.

We now wish to analyse these conserved charges from the point of view of the three-dimensional theory defined on V, and to clarify their transformation properties under the action of the three-dimensional duality group $\mathfrak{G}$. In consequence of the invariance of the three-dimensional action under this group, there exists an associated Noether current in three dimensions. Indeed, the equations of motion (2.4) can be rewritten as

$$d\star\mathcal{V}P\mathcal{V}^{-1} = 0 \ . \tag{2.8}$$

Therefore, the $\mathfrak{g}$-valued Noether current is $\star\mathcal{V}P\mathcal{V}^{-1}$. Since the three-dimensional theory is Euclidean, we cannot properly talk about a conserved charge. Nevertheless, since $\star\mathcal{V}P\mathcal{V}^{-1}$ is d-closed, the integral of this 2-form on a given homology cycle does not depend on the representative of the cycle. As a result, for stationary solutions, the integral of this three-dimensional current over any spacelike closed surface, containing in its interior all the singularities and topologically non-trivial subspaces of the solution, defines a $\mathfrak{g}$-valued charge matrix $\mathscr{C}$:

$$\mathscr{C} \equiv \frac{1}{4\pi}\int_{\partial V} \star\mathcal{V}P\mathcal{V}^{-1} \ . \tag{2.9}$$

This transforms in the adjoint representation of $\mathfrak{G}$ by the standard non-linear action and it can easily be computed by looking at the asymptotic value of the current if (as we assume to be the case)

$$P = \mathscr{C}\,\frac{dr}{r^2} + \mathcal{O}(r^{-2}) \ . \tag{2.10}$$

For asymptotically flat solutions, $\mathcal{V}$ goes to the identity matrix asymptotically and the charge matrix $\mathscr{C}$ in that case is given by the asymptotic value of the one-form P. $\mathscr{C}$ then lies in $\mathfrak{g}\ominus\mathfrak{h}^*$ and can thus be decomposed into three irreducible representations with respect to $\mathfrak{so}(2)\oplus\mathfrak{h}_4$ according to

$$\mathfrak{g}\ominus\mathfrak{h}^* \cong \big(\mathfrak{sl}(2,\mathbb{R})\ominus\mathfrak{so}(2)\big)\oplus\mathfrak{l}_4\oplus\big(\mathfrak{g}_4\ominus\mathfrak{h}_4\big) \tag{2.11}$$

We stress once again that the metric induced by the Cartan-Killing metric of $\mathfrak{g}$ on the coset (2.11) is positive definite on the first and last summand, and negative definite on $\mathfrak{l}_4$.

The decomposition (2.11) is in precise accord with the structure of the conserved charges in four dimensions as described above. Namely, the computation of $\mathscr{C}$ permits one to identify its $\mathfrak{sl}(2,\mathbb{R}) \ominus \mathfrak{so}(2)$ component as the Komar mass and the Komar NUT charge, and its $\mathfrak{l}_4$ components with the electromagnetic charges. The remaining $\mathfrak{g}_4 \ominus \mathfrak{h}_4$ charges come from the $\mathfrak{G}_4$ Noether current of the original four-dimensional theory, which transforms in the adjoint of $\mathfrak{G}_4$. For a stationary solution, $\mathcal{L}_\kappa J_3 = 0$ and $i_\kappa J_3$ then defines a conserved two-form which is furthermore manifestly invariant and horizontal with respect to the timelike isometry. Although J_3 in general is not a local function of the fields and their four-dimensional derivatives, $i_\kappa J_3$ can be written in terms of the pull-backs of the scalar fields of the three-dimensional model for stationary field configurations. One thus obtains that the integral of the pull-back

$$\frac{1}{4\pi}\int_\Sigma s^* i_\kappa J_3 \tag{2.12}$$

on any homology two-cycle Σ of V, does not depend on the representative of that cycle. An important fact is that *the scalar charges, that is, the $\mathfrak{g}_4 \ominus \mathfrak{h}_4$ components of $\mathscr{C}$, will not constitute independent integration parameters.* This was demonstrated in full generality in [1]. We will see that it is natural to impose characteristic equations on the charges, with the consequence that the scalar charges become functions of the gravitational and electromagnetic charges in the case of pure supergravity theories. We note also that the contribution of the angular momentum in (2.10) is subleading (that is, it belongs to the $\mathcal{O}(r^{-2})$ part of (2.10)); hence the conserved charge $\mathscr{C}$ will be insensitive to the angular momentum parameter a.

Defining the usual generators of $\mathfrak{sl}(2,\mathbb{R})$, $\boldsymbol{h}, \boldsymbol{e}$ and $\boldsymbol{f}$ by

$$[\boldsymbol{h},\boldsymbol{e}] = 2\boldsymbol{e} \qquad [\boldsymbol{h},\boldsymbol{f}] = -2\boldsymbol{f} \qquad [\boldsymbol{e},\boldsymbol{f}] = \boldsymbol{h} \tag{2.13}$$

we can summarise what has been said above in the equation

$$\star\, \mathcal{V}P\mathcal{V}^{-1} = 4s^* \star K\boldsymbol{h} - 4s^* K(\boldsymbol{e}+\boldsymbol{f}) + s^* \star \mathcal{F} + s^* F + s^* i_\kappa J_3 + \mathcal{O}(r^{-2}) \tag{2.14}$$

where the electromagnetic current $s^* \star \mathcal{F} + s^* F$, which transforms under $\mathfrak{G}_4$ in the representation $\mathfrak{l}_4$, is understood to be valued in the corresponding generators of $\mathfrak{G}$ with the appropriate normalisation. For example, in $\mathcal{N} = 8$ supergravity the 28 Maxwell fields F^{ij} transform under $\mathrm{SO}(8) \subset \mathrm{SU}(8)$ as antisymmetric tensors. The compact generators of $\mathfrak{e}_{8(8)} \ominus \mathfrak{so}^*(16)$ transform in the **56** of $E_{7(7)}$. They are conveniently represented by a complex antisymmetric tensor $\boldsymbol{Z}^{ij}$ of SU(8) and its Hermitean conjugate $\boldsymbol{Z}_{ij}$. Then

$$\star\, \mathcal{F} = \star\mathcal{F}_{ij}\boldsymbol{Z}^{ij} - \star\mathcal{F}^{ij}\boldsymbol{Z}_{ij} \qquad F = iF_{ij}\boldsymbol{Z}^{ij} + iF^{ij}\boldsymbol{Z}_{ij}\ . \tag{2.15}$$

Note that only the sum $\star\mathcal{F} + F$ transforms covariantly under the action of $E_{7(7)}$.

The charge matrix $\mathscr{C}$ is associated to "instantons" of the three-dimensional Euclidean theory. The single-point instantons correspond to single-particle like solutions of the four-dimensional theory. Naively, one would thus expect these solutions to appear in multiplets transforming in the linear representation of $\mathfrak{H}^*$ defined by $\mathfrak{g} \ominus \mathfrak{h}^*$. However, matters are not so simple, because the charge matrix $\mathscr{C}$ is restricted to satisfy $\mathfrak{H}^*$ invariant constraints in general, so that the number of independent parameters describing these solutions is much smaller – as was to be anticipated in view of the dependence of the charges associated to the four-dimensional scalar fields on the gravitational and electromagnetic charges. This is because the charges parametrising the solutions are the conserved charges *in four dimensions*, that is the mass, the NUT charge and the electromagnetic charges. This, in turn, is due to the fact that the particle-like solutions are supported by vector fields through Gauss's law. A useful analogy here is that of a free particle in Minkowski space. When the momentum of this particle is timelike, it can be rotated to the rest frame. Here, the role of momentum is played by the charge matrix, while the non-compact group $\mathfrak{H}^*$ plays the role of the Lorentz group. The electromagnetic charges belong to the $\mathfrak{l}_4$ representation of $\mathfrak{G}_4$, just like the non-compact generators of $\mathfrak{h}^*$. The action of these generators on the Maxwell charges is linear in the scalar and the gravity charges, in such a way that for a non-zero value of $m^2 + n^2$ one can always find a generator that acts on the Maxwell charge as a shift parallel to it. This generator of $\mathfrak{h}^*$ defines an $\mathrm{SO}(1,1)$ subgroup of $\mathfrak{H}^*$ which mixes the electromagnetic charges with the others. For any charge matrix satisfying $\mathrm{Tr}\, \mathscr{C}^2 > 0$ the action of this abelian subgroup of $\mathfrak{H}^*$ permits one to cancel the electromagnetic charges. It then follows from the five-graded decomposition of $\mathfrak{g}$ that one can find an element of the compact subgroup of the Ehlers group that cancels the NUT charge without modifying the electromagnetic and the scalar charges. It has been proven in [1] that a static solution without electromagnetic charges will have singularities outside the horizons if the scalar fields are not constant throughout spacetime. In this way, a theorem was proved that all static solutions regular outside the horizon with a charge matrix satisfying $\mathrm{Tr}\, \mathscr{C}^2 > 0$ lie on the $\mathfrak{H}^*$-orbit of the Schwarzschild solution. This also led to a generalisation of Mazur's theorem, obtaining that all non-extremal axisymmetric stationary and asymptotically Minkowskian black holes lie on the $\mathfrak{H}^*$-orbit of the Kerr solution (with some angular momentum parameter a).[6]

Although the Mazur proof is more difficult to generalise to the case of asymptotically Taub–NUT solutions, it is reasonable to conjecture that all non-extremal axisymmetric stationary particle-like solutions lie on the $\mathfrak{H}^*$-orbit of some Kerr solution.

It follows, as a corollary, that any $\mathfrak{H}^*$ invariant equation satisfied by the charge matrix $\mathscr{C}$ of a Kerr solution is also satisfied by the charge matrix of any non-extremal axisymmetric stationary particle-like solution. Although there is no general proof that all the extremal axisymmetric stationary particle-like solutions can be obtained by taking the appropriate limit of a non-extremal solution, so far all known such solutions can be obtained in this way. By continuity, any $\mathfrak{H}^*$ invariant equation satisfied by the charge matrix $\mathscr{C}$ is also valid for such extremal solutions. Using Weyl coordinates [28], the coset representative $\mathcal{V}$ associated to the Schwarzschild solution with mass m and its associated charge can be

[6]For the reader's convenience we recall that Mazur's theorem states that an asymptotically Minkowskian axisymmetric stationary non-extremal black hole solution with a non-degenerate horizon is uniquely determined by its mass, its angular momentum and its electromagnetic charges [27].

written in terms of the non-compact generator $\boldsymbol{h}$ of $\mathfrak{sl}(2,\mathbb{R})$ only, viz.

$$\mathcal{V} = \exp\left(\frac{1}{2}\ln\frac{r-m}{r+m}\,\boldsymbol{h}\right) \qquad \Rightarrow \qquad \mathscr{C} = m\boldsymbol{h} \tag{2.16}$$

where we have used (2.9). According to the five-graded decomposition (2.1), the generator $\boldsymbol{h}$ in the adjoint representation acts as the diagonal matrix diag $[2,1,0,-1,-2]$, where 1 is the identity on $\mathfrak{l}_4$ and 0 acts on $\mathfrak{g}_4 \oplus \{\boldsymbol{h}\}$. This implies that

$$\mathrm{ad}_{\boldsymbol{h}}{}^5 = 5\mathrm{ad}_{\boldsymbol{h}}{}^3 - 4\,\mathrm{ad}_{\boldsymbol{h}} \ . \tag{2.17}$$

The four-dimensional theories leading to coset models associated to simple groups $\mathfrak{G}$ after timelike dimensional reduction have been classified in [1]. They correspond to models for which the four-dimensional scalars parametrise a symmetric space whose isometry group acts non-trivially on the vector fields. In particular, the list of [1] includes two theories for which the three-dimensional duality group is a real form of E_8, namely $\mathcal{N}=8$ supergravity [4], and the exceptional 'magic' $\mathcal{N}=2$ supergravity [29] with real forms $E_{8(8)}$ and $E_{8(-24)}$, respectively. Since the fundamental representation of E_8 is the adjoint representation, we have, for these two theories,

$$\boldsymbol{h}^5 = 5\boldsymbol{h}^3 - 4\boldsymbol{h} \ . \tag{2.18}$$

However, $\boldsymbol{h}$ turns out to satisfy a lower order polynomial equation in general. Indeed, for all the other groups listed in [1], the fundamental representation of $\mathfrak{G}$ admits a three-graded decomposition with respect to the generator $\boldsymbol{h}$, in such a way that the latter takes the form diag $[1,0,-1]$. The three-graded decomposition of the groups listed in [1] is displayed in appendix A. It follows that in these cases one has the stronger relation

$$\boldsymbol{h}^3 = \boldsymbol{h} \ . \tag{2.19}$$

We then define the BPS parameter c^2 by

$$c^2 \equiv \frac{1}{k}\,\mathrm{Tr}\,\mathscr{C}^2 \tag{2.20}$$

with $k \equiv \mathrm{Tr}\,\boldsymbol{h}^2 > 0$, where the normalisation is chosen such that $c^2 = m^2$ for the Schwarzschild solution. Owing to the indefinite metric on the coset space $\mathfrak{g} \ominus \mathfrak{h}^*$, the trace $\mathrm{Tr}\,\mathscr{C}^2$ and thus the square of the BPS parameter c^2 can assume either sign.[7] However, negative values of c^2 correspond to hyper-extremal solutions which we will not consider (such as e.g. the Reissner-Nordström solution with $c^2 = m^2 - e^2 < 0$, which has a naked singularity). Hence, the BPS parameter will always be assumed to be non-negative in the following. Equation (2.18) then implies that for any solution in the $\mathfrak{H}^*$-orbit of the Kerr solution, the charge matrix $\mathscr{C}$ satisfies

$$\mathscr{C}^5 = 5c^2\mathscr{C}^3 - 4c^4\mathscr{C} \tag{2.21}$$

[7]This is in contradistinction to spacelike reductions, for which the metric on the coset is positively defined, whence $\mathrm{Tr}\,\mathscr{C}^2 = 0$ would imply $\mathscr{C} = 0$. We thus recover the well-known result that, in order for BPS solutions to exist, the Killing vector must be non-spacelike [3].

For all but two exceptional cases with $E_{8(8)}$ and $E_{8(-24)}$, we have the stronger constraint

$$\mathscr{C}^3 = c^2 \mathscr{C} \tag{2.22}$$

from (2.19), in which case the fundamental representation admits a three-graded decomposition. Then, using the theorem of [1], it follows that these equations are satisfied by the charge matrix of any asymptotically flat non-extremal axisymmetric stationary single-particle solution. Furthermore, it follows that *non-rotating extremal solutions (like the BPS solutions), for which $c = 0$, are characterised by nilpotent charge matrices $\mathscr{C}$.* Note however that the BPS parameter is non-zero for extremal rotating solutions. The extremality parameter $\varkappa$ is defined as

$$\varkappa \equiv \sqrt{c^2 - a^2} \tag{2.23}$$

where a is the angular momentum by unit of mass.[8] For an asymptotically Taub–NUT black hole, the extremality parameter is equal to the product of the horizon area and the surface gravity divided by a factor of 4π. Neither the horizon area nor the surface gravity is left invariant by the action of $\mathfrak{H}^*$, but nevertheless $\varkappa$ is an invariant.

The current $\star \mathcal{V} P \mathcal{V}^{-1}$ is the representative of a cohomology class of V, and as such it defines a linear map from the second homology group of V to $\mathfrak{g}$.

$$\begin{array}{rcl} \star\, \mathcal{V} P \mathcal{V}^{-1} \;:\; H^2(V) & \longrightarrow & \mathfrak{g} \\ \Sigma & \longrightarrow & \mathscr{C}_{|\Sigma}\ . \end{array} \tag{2.24}$$

The algebraic structure of $\mathfrak{g}$ then permits one to define a non-linear map from $H^2(V)$ into the universal enveloping algebra of $\mathfrak{g}$ for any polynomial Υ as follows

$$\begin{array}{rcl} \Upsilon \;:\; H^2(V) & \longrightarrow & \mathrm{U}(\mathfrak{g}) \\ \Sigma & \longrightarrow & \Upsilon(\mathscr{C}_{|\Sigma})\ . \end{array} \tag{2.25}$$

For any stationary asymptotically flat solution regular outside the horizon of any non-exceptional model of the list [1], we conjecture that this map vanishes identically on $H^2(V)$ for $\Upsilon = \mathscr{C}^3 - c^2\mathscr{C}$, and moreover that it vanishes for $\Upsilon = \mathscr{C}^5 - 5c^2\mathscr{C}^3 + 4c^4\mathscr{C}$ in the exceptional cases of $\mathcal{N} = 8$ supergravity and the exceptional $\mathcal{N} = 2$ magic supergravity.

2.2 Supergravity and BPS conditions

When considering stationary solutions in supergravity theories, the Euclidean three-dimensional point of view is very convenient for obtaining an understanding of the BPS properties of stationary solutions. $\mathcal{N}$-extended supergravity in four dimensions admits $\mathrm{U}(\mathcal{N})$ as an R symmetry group for $\mathcal{N} < 8$, and $\mathrm{SU}(8)$ for $\mathcal{N} = 8$. Upon dimensional reduction from four to three dimensions, the compact R symmetry $\mathrm{U}(\mathcal{N})$ is enlarged to $\mathrm{SO}(2\mathcal{N})$ if the Killing vector is spacelike [7], and to the group $\mathrm{SO}^*(2\mathcal{N})$ (non-compact for $\mathcal{N} > 1$) if it is timelike [1]. It is the latter case that is relevant to the stationary solutions

[8] Note that the definition of angular momentum is slightly more subtle in asymptotically Taub–NUT spacetimes, nevertheless one can define it unambiguously by requiring the corresponding Komar integral to be independent of the local section of $\mathrm{U}(1) \to M_+ \to V$ [25].

considered here. In this case, the group of automorphisms of the $2\mathcal{N}$-extended superalgebra in three dimensions is the product of the three-dimensional rotation group SU(2) and the R symmetry group $\mathrm{Spin}^*(2\mathcal{N})$.

The fields of pure $\mathcal{N}$-extended supergravity are in one-to-one correspondence with the p-form representations of $\mathrm{U}(\mathcal{N})$, where p is even for the boson fields, and odd for the fermionic fields. For $\mathcal{N} = 8$, there is no U(1) factor, and the p-form representations are related by duality to the complex conjugates of the $(8-p)$-form representations, while scalar fields are complex self-dual (i.e. pseudo-real). As we will explain, there is a similar pattern for the conserved charges of the stationary solutions: the mass and the NUT charge correspond to the trivial representation of $\mathrm{SU}(\mathcal{N})$ while the electromagnetic charges correspond to the 2-form and the 6-form representations of $\mathrm{SU}(\mathcal{N})$ and the scalar charges correspond to the 4-form representation of $\mathrm{SU}(\mathcal{N})$. After a timelike dimensional reduction, these then combine to form the full charge matrix $\mathscr{C}$ of pure $\mathcal{N}$-extended supergravity, which will be shown to be equivalently described by a state $|\mathscr{C}\rangle$ transforming in the Weyl spinor representation of $\mathrm{Spin}^*(2\mathcal{N})$. Likewise, and in analogy with the spacelike reduction of [7], the bosonic and fermionic fields are assigned to spinor representations of $\mathrm{Spin}^*(2\mathcal{N})$, and are transformed into one another by the action of $2\mathcal{N}$ extended supersymmetry, with the supersymmetry parameter belonging to the (pseudo-real) vector representation of $\mathrm{SO}^*(2\mathcal{N})$.

Following [7] one can now in principle classify all possible locally supersymmetric theories systematically by studying the restrictions that supersymmetry imposes on the target space geometries. Here we will not work out the complete Lagrangians, but will concentrate on the relevant supersymmetry variations. Furthermore, we will limit attention to the smaller class of theories obtainable by dimensional reduction from four dimensional supergravities, and whose scalar sectors are governed by irreducible symmetric spaces. A list of such theories can be obtained by matching the tables of [1] with previous results on spacelike reductions of [7]. The Kähler symmetric spaces can be found in [30], and the special Kähler symmetric spaces have been classified in [31].

For $\mathcal{N} = 1$ supergravity theories, the internal symmetry group is the product of $\mathrm{Spin}^*(2) \cong \mathrm{U}(1)$ and a group associated to the matter content of the theory. A list of the relevant theories is given in table 1 below, with the number of vector and scalar supermultiplets in four dimensions given in the third and fourth columns, respectively.

For $\mathcal{N} = 2$, the internal symmetry is the product of $\mathrm{Spin}^*(4)$ with a group associated to the matter content of the theory (vector multiplets or hypermultiplets). Now, $\mathrm{Spin}^*(4) \cong \mathrm{SU}(1,1) \times \mathrm{SU}(2)$, where the SU(2) factor acts only on the scalar fields belonging to hypermultiplets, and on the fermions. The theories that can be analysed within the present framework have vector multiplets but no hypermultiplets in four dimensions (the number is given in the third column of the table below). These models are displayed in table 2.

For $\mathcal{N} \geq 3$, the possible supergravity theories are much more constrained, and the target spaces must be symmetric spaces.[9] When $\mathcal{N} = 3, 4$, we can still couple in an arbitrary number of matter multiplets, whereas for $\mathcal{N} \geq 5$ the theories are uniquely determined.

[9]Whereas this is not true for the $\mathcal{N} \leq 2$ theories, cf. [7].

$\mathfrak{G}/\mathfrak{H}^*$	$\mathfrak{G}_4/\mathfrak{H}_4$	Vector	Scalar
$\frac{\mathrm{SU}(1+m,1+n)}{\mathrm{U}(1)\times\mathrm{SU}(m,1)\times\mathrm{SU}(1,n)}$	$\frac{\mathrm{U}(m,n)}{\mathrm{U}(m)\times\mathrm{U}(n)}$	$m+n$	mn
$\frac{\mathrm{SO}^*(4+2n)}{\mathrm{U}(1)\times\mathrm{SU}(2,n)}$	$\frac{\mathrm{SO}^*(2n)\times\mathrm{SU}(2)}{\mathrm{U}(n)\times\mathrm{SU}(2)}$	$2n$	$\frac{n(n-1)}{2}$
$\frac{\mathrm{Sp}(2+2n,\mathbb{R})}{\mathrm{U}(1)\times\mathrm{SU}(1,n)}$	$\frac{\mathrm{Sp}(2n,\mathbb{R})}{\mathrm{U}(n)}$	n	$\frac{n(n+1)}{2}$
$\frac{E_{7(-25)}}{\mathrm{U}(1)\times E_{6(-14)}}$	$\frac{\mathrm{SO}(2,10)}{\mathrm{SO}(2)\times\mathrm{SO}(10)}$	16	10

Table 1. Irreducible homogenous spaces of $\mathcal{N}=1$ supergravity

$\mathfrak{G}/\mathfrak{H}^*$	$\mathfrak{G}_4/\mathfrak{H}_4$	Vector
$\frac{\mathrm{SU}(2,1+n)}{\mathrm{SU}(1,1)\times\mathrm{U}(1,n)}$	$\frac{\mathrm{U}(1,n)}{\mathrm{U}(1)\times\mathrm{U}(n)}$	n
$\frac{\mathrm{Spin}(4,2+n)}{\mathrm{SU}(1,1)\times\mathrm{SU}(1,1)\times\mathrm{Spin}(2,n)}$	$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\times\frac{\mathrm{SO}(2,n)}{\mathrm{U}(1)\times\mathrm{SO}(n)}$	$1+n$
$\frac{G_{2(2)}}{\mathrm{SU}(1,1)\times\mathrm{SU}(1,1)}$	$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$	1
$\frac{F_{4(4)}}{\mathrm{SU}(1,1)\times\mathrm{Sp}(6,\mathbb{R})}$	$\frac{\mathrm{Sp}(6,\mathbb{R})}{\mathrm{U}(1)\times\mathrm{SU}(3)}$	6
$\frac{E_{6(6)}}{\mathrm{SU}(1,1)\times\mathrm{SU}(3,3)}$	$\frac{\mathrm{SU}(3,3)}{\mathrm{U}(1)\times\mathrm{SU}(3)\times\mathrm{SU}(3)}$	9
$\frac{E_{7(-5)}}{\mathrm{SU}(1,1)\times\mathrm{SO}^*(12)}$	$\frac{\mathrm{SO}^*(12)}{\mathrm{U}(1)\times\mathrm{SU}(6)}$	15
$\frac{E_{8(-24)}}{\mathrm{SU}(1,1)\times E_{7(-25)}}$	$\frac{E_{7(-25)}}{\mathrm{U}(1)\times E_{6(-78)}}$	27

Table 2. Irreducible homogenous spaces of $\mathcal{N}=2$ supergravity.

The complete list is given in table 3 below. Note that we need to invoke the low rank isomorphisms $\mathrm{Spin}^*(6)\cong\mathrm{SU}(3,1)$ (for $\mathcal{N}=3$) and $\mathrm{Spin}^*(8)\cong\mathrm{Spin}(6,2)$ (for $\mathcal{N}=4$), respectively, in order to match the tables with the general theory.

Let us now discuss the supersymmetry variations relevant to the BPS analysis in more detail. For the Lorentzian case (i.e. for a spacelike reduction), the relevant (massless) supermultiplets were already described and studied in [7]. As shown there, these superalgebras and their (massless) representations can be completely characterised in terms of the *real Clifford algebras*

$$\{\Gamma^I,\Gamma^J\}=2\delta^{IJ}\qquad\text{for } I,J,\ldots=1,\ldots,2\mathcal{N}\tag{2.26}$$

Here we must perform a similar analysis, but with $\mathrm{SO}(2\mathcal{N})$ replaced by $\mathrm{SO}^*(2\mathcal{N})$ whose maximal compact subgroup is $\mathrm{U}(\mathcal{N})$ (by definition). Since the $2\mathcal{N}$-extended Minkowskian superalgebra thus admits a Clifford algebra construction from $\mathrm{Cl}(2\mathcal{N},\mathbb{R})$, we will look for an analogous construction for $\mathrm{SO}^*(2\mathcal{N})$ making use of the complex Clifford algebra $\mathrm{Cl}(\mathcal{N},\mathbb{C})$. Because the group $\mathrm{Spin}^*(2\mathcal{N})$ and its irreducible spinorial representations are

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$\mathcal{N}$	$\mathfrak{G}/\mathfrak{H}^*$	$\mathfrak{G}_4/\mathfrak{H}_4$	Vector
3	$\frac{\mathrm{SU}(4,1+n)}{\mathrm{SU}(3,1)\times\mathrm{U}(1,n)}$	$\frac{\mathrm{U}(3,n)}{\mathrm{U}(3)\times\mathrm{U}(n)}$	n
4	$\frac{\mathrm{SO}(8,2+n)}{\mathrm{SO}(6,2)\times\mathrm{SO}(2,n)}$	$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\times\frac{\mathrm{Spin}(6,n)}{\mathrm{SU}(4)\times\mathrm{Spin}(n)}$	n
5	$\frac{E_{6(-14)}}{\mathrm{Spin}^*(10)\times\mathrm{U}(1)}$	$\frac{\mathrm{SU}(5,1)}{\mathrm{U}(5)}$	0
6	$\frac{E_{7(-5)}}{\mathrm{Spin}^*(12)\times\mathrm{SU}(1,1)}$	$\frac{\mathrm{Spin}^*(12)}{\mathrm{U}(6)}$	0
8	$\frac{E_{8(8)}}{\mathrm{Spin}^*(16)}$	$\frac{E_{7(7)}}{\mathrm{SU}(8)}$	0

Table 3. Homogenous spaces of $\mathcal{N} \geq 3$ supergravity.

perhaps less familiar, we summarise some relevant results in appendix B. Besides the use of manifestly U($\mathcal{N}$) covariant notation, the crucial tool here is the use of fermionic oscillators defined by (see e.g. [32] for a pedagogical introduction)

$$a_i := \frac{1}{2}\Big(\Gamma_{2i-1} + i\Gamma_{2i}\Big) \quad , \qquad a^i \equiv (a_i)^\dagger = \frac{1}{2}\Big(\Gamma_{2i-1} - i\Gamma_{2i}\Big) \tag{2.27}$$

for $i, j, \dots = 1, \dots, \mathcal{N}$. These obey the standard anticommutation relations

$$\{a_i, a_j\} = \{a^i, a^j\} = 0 \quad , \qquad \{a_i, a^j\} = \delta_i^j \ . \tag{2.28}$$

As we will see this formalism greatly facilitates the analysis of the BPS conditions.

Making use of the fermionic oscillators introduced above we will thus express the various fields in the spinor basis generated by the creation operators a^i acting on a 'vacuum state' $|0\rangle$ (which is annihilated by all the a_i). Accordingly, for $\mathcal{N} = 5, 6, 8$ the coset components $P_\mu dx^\mu$ of the Cartan form are represented by the state

$$|P_\mu\rangle = \Big(P^{(0)}_\mu + P^{(2)}_{\mu\, ij} a^i a^j + P^{(4)}_{\mu\, ijkl} a^i a^j a^k a^l + \dots\Big)|0\rangle \ . \tag{2.29}$$

For $\mathcal{N} \leq 4$, an arbitrary number of matter multiplets can be coupled and therefore the state $|P_\mu\rangle$ carries an extra label $\mathcal{A}$ to count the matter multiplets. Inspecting the $\mathfrak{H}^*$ groups in the tables we see this extra label is an SO$(2, n)$ index for $\mathcal{N} = 4$ (cf. our discussion of matter-coupled $\mathcal{N} = 4$ supergravities in section 6), an SU$(1, n)$ index for $\mathcal{N} = 3$, and so on. Furthermore, the state $|P_\mu\rangle$, or the states $|P_\mu, \mathcal{A}\rangle$, in principle must satisfy an irreducibility (reality) constraint as explained in appendix B. However, when the group $\mathfrak{H}^*$ contains an extra U(1) factor besides the R symmetry group Spin$^*(2\mathcal{N})$, the representation (2.29) becomes complexified, so we only need to impose a reality constraint when no U(1) is available, such as for instance $\mathcal{N} = 2$ supergravity with exceptional $\mathfrak{G}$ or $\mathfrak{G} = \mathrm{Spin}(4, 2+n)$. The case $\mathcal{N} = 6$ can be obtained by a consistent truncation from $\mathcal{N} = 8$ (see section 2.2); for the latter, there is again no U(1) factor in $\mathfrak{H}^*$, and we need to require $|P\rangle = \star|P\rangle$, which expresses the well known self-duality of the $\mathcal{N} = 8$ multiplet. Similarly, the physical fermions are represented by the anti-chiral state

$$|\chi\rangle_\alpha = \Big(\psi_{\alpha\, i} a^i + \chi_{\alpha\, ijk} a^i a^j a^k + \dots\Big)|0\rangle \tag{2.30}$$

(We use the letter ψ_i for degree-one components, because these originate from the 4-dimensional gravitinos, while the χ_{ijk} originate from the 4-dimensional spin-$\frac{1}{2}$ fermions.) Again, this representation must satisfy an irreducibility constraint for $\mathcal{N}=8$, namely[10]

$$(\star|\chi\rangle)^\alpha = \varepsilon^{\alpha\beta}|\chi\rangle_\beta\ . \tag{2.31}$$

The $\varepsilon_{\alpha\beta}$ here is necessary because the anti-chiral representation of $\mathrm{Spin}^*(4M)$ is *pseudo-real*, with twice as many components as the *real* chiral spinor. It is thus the additional spatial $\mathrm{SU}(2)$ symmetry that restores the boson fermion balance required by supersymmetry. For theories with an arbitrary number of matter multiplets the fermionic state acquires an extra label, just like the bosonic state.

While the $\mathrm{U}(\mathcal{N})$ transformation properties of these states are manifest, they transform as follows under the non-compact generators of $\mathrm{Spin}^*(2\mathcal{N})$ (cf. appendix B):

$$\delta\,|P_\mu\rangle = \frac{1}{2}\Big(\Lambda^{ij}a_i a_j - \Lambda_{ij}a^i a^j\Big)\,|P_\mu\rangle \quad , \quad \delta\,|\chi\rangle_\alpha = \frac{1}{2}\Big(\Lambda^{ij}a_i a_j - \Lambda_{ij}a^i a^j\Big)\,|\chi\rangle_\alpha \tag{2.32}$$

where for $\mathcal{N}\le 4$ we suppress the extra index $\mathcal{A}$ for simplicity. On the other hand, the three-dimensional gravitinos ψ^i_α and the supersymmetry parameters ϵ^i_α together with their complex conjugates $\psi_i^\alpha \equiv (\psi^i_\alpha)^*$ and $\epsilon_i^\alpha \equiv (\epsilon^i_\alpha)^*$ transform in the pseudo-real vector representation of $\mathrm{SO}^*(2\mathcal{N})$, that is[11]

$$\delta\epsilon^i_\alpha = \Lambda^i{}_j\epsilon^j_\alpha + \Lambda^{ij}\varepsilon_{\alpha\beta}\epsilon^\beta_j \quad , \qquad \delta\epsilon_i^\alpha = \Lambda_i{}^j\epsilon_j^\alpha + \Lambda_{ij}\varepsilon^{\alpha\beta}\epsilon^j_\beta \tag{2.33}$$

and similarly for the gravitinos. The commutator of two $\mathrm{Spin}^*(2\mathcal{N})$ transformations with parameters Λ_1 and Λ_2 gives a new transformation with parameters

$$\begin{aligned}
\Lambda_{12}{}^i_j &= \Lambda_1{}^i{}_k\Lambda_2{}^k{}_j - \Lambda_2{}^i{}_k\Lambda_1{}^k{}_j + \Lambda_1{}^{ik}\Lambda_{2\,jk} - \Lambda_2{}^{ik}\Lambda_{1\,ik}\\
\Lambda_{12}^{ij} &= -2\Lambda_1{}^{[i}{}_k\Lambda_2{}^{j]k} + 2\Lambda_2{}^{[i}{}_k\Lambda_1{}^{j]k}\ .
\end{aligned} \tag{2.34}$$

With this notation, the supersymmetry transformations of the fermions read

$$\delta\psi^i_\alpha = d_{\omega+Q}\epsilon^i_\alpha \quad , \qquad \delta\psi_i^\alpha = d_{\omega+Q}\epsilon_i^\alpha \tag{2.35}$$

for the gravitino components, and

$$\delta\,|\chi\rangle_\alpha = e^\mu_a\sigma^a{}_\alpha{}^\beta\,\Big(\epsilon^i_\beta\,a_i + \varepsilon_{\beta\gamma}\epsilon^\gamma_i a^i\Big)\,|P_\mu\rangle \tag{2.36}$$

[10] Note that $\star$ being an *anti*-pseudo-involution, it raises the $\mathrm{SU}(2)$ index by complex conjugation.

[11] In the Minkowski case, the fundamental representation of $\mathrm{SL}(2,\mathbb{R})$ is real, and the transformation is simply $\delta\epsilon^I_\alpha = \Lambda^{IJ}\epsilon^J_\alpha$ (recall that $I,J,\dots = 1,\dots,2\mathcal{N}$). In the complex $\mathrm{U}(\mathcal{N})$ basis, this becomes

$$\delta\epsilon^i_\alpha = \Lambda^i{}_j\epsilon^j_\alpha + \Lambda^{ij}\epsilon_{\alpha\,j}.$$

The $\varepsilon_{\alpha\beta}$ in (2.33) thus plays the role of an imaginary unit.

for the propagating fermions, where $d_{\omega+Q}$ is the covariant exterior differential with respect to the SU(2) spin-connection ω and the $\mathfrak{H}^*$ connection Q coming from the scalar fields. We note that for $\mathcal{N}=8$ this formula is consistent with the representation constraint, that is, $(\star\delta|\chi\rangle)^\alpha = \varepsilon^{\alpha\beta}\delta|\chi\rangle_\beta$. Using the above definitions and the formulas from appendix B (and, more specifically, the fact that conjugation for a Spin*$(2\mathcal{N})$ spinor involves the matrix β), it is straightforward to compute the conjugate spinor supersymmetry transformations:

$$\delta\,\langle\chi|^\alpha = -e^\mu_a\sigma^{a\,\alpha}{}_\beta\,\langle P_\mu|\left(\epsilon^\beta_i\,a^i + \varepsilon^{\beta\gamma}\epsilon^i_\gamma a_i\right)\,. \tag{2.37}$$

The integrability condition for a supersymmetry transformation with parameter ϵ to preserve the vanishing of the gravitino fields is given by the algebraic equation

$$\delta\psi^\alpha_i = 0 \quad\Rightarrow\qquad (\not{R} + dQ + Q^2)\epsilon = 0 \tag{2.38}$$

for the curvature 2-forms $\not{R} \equiv \frac{1}{4}R^{ab}{}_{\mu\nu}\sigma_{ab}dx^\mu\wedge dx^\nu$ and $dQ+Q^2$, valued in $\mathfrak{su}(2)$ and $\mathfrak{so}^*(2\mathcal{N})$, respectively. In three dimensions, the curvature 2-form $\not{R}$ is expressible[12] in terms of the Ricci tensor R_{ab} by

$$\not{R} = \frac{1}{2}\sigma^{ab}\Big(e_a\wedge e^c\,R_{cb} - e_b\wedge e^c\,R_{ac} - \frac{1}{2}e_a\wedge e_b\,R\Big)\,. \tag{2.39}$$

The equations of motion (2.5) give furthermore that

$$\not{R} = \frac{1}{2}\sigma^{ab}\Big(e_a\wedge dx^\mu e^\nu_b - e_b\wedge dx^\mu e^\nu_a - \frac{1}{2}e_a\wedge e_b g^{\mu\nu}\Big)\mathrm{Tr}\;P_\mu P_\nu\;. \tag{2.40}$$

Then, using the Bianchi identity (2.3), one can rewrite the integrability condition in terms of the one-form P only,

$$\left[\frac{1}{2}\sigma^{ab}\Big(e_a\wedge dx^\mu e^\nu_b - e_b\wedge dx^\mu e^\nu_a - \frac{1}{2}e_a\wedge e_b g^{\mu\nu}\Big)\mathrm{Tr}\;P_\mu P_\nu - P\wedge P\right]\epsilon = 0\;. \tag{2.41}$$

For asymptotically flat solutions, P goes to zero as in (2.10) for $r\to+\infty$, and the leading order part of this equation is given by

$$\frac{1}{2r^4}\sigma^{ab}\Big(\delta^3_a dr\wedge e_b - \delta^3_b dr\wedge e_a - \frac{1}{2}e_a\wedge e_b\Big)\left(\mathrm{Tr}\;\mathscr{C}^2\right)\epsilon = \mathcal{O}(r^{-3})\,, \tag{2.42}$$

where $e^3\sim dr+\mathcal{O}(1)$. In this way, we arrive at the condition

$$\mathrm{Tr}\;\mathscr{C}^2 = 0 \quad\Leftrightarrow\quad c^2 = 0\;. \tag{2.43}$$

If this equation is satisfied, one can then integrate the first order equation for the Killing spinors following from the supersymmetry variations of the gravitinos, thus justifying the designation of c^2 as the 'BPS parameter'. We stress once more that (2.43) does *not* imply $\mathscr{C}=0$. For asymptotically Minkowski solutions (that is, without NUT charge), we have

[12] The formula is

$$R_{\mu\nu ab} = 4e_{[\mu[a}R_{b]\nu]} - e_{\mu[a}e_{\nu|b]}R.$$

checked that, for the pure $\mathcal{N} \leq 5$ supergravities, c^2 is indeed proportional to the determinant of the Bogomolny matrix[13] (this claim will be proved in section 3.3). This is no longer true for $\mathcal{N} = 6$ and $\mathcal{N} = 8$. From equation (2.21), one then deduces that *the charge matrix is nilpotent for BPS solutions*. More precisely, we have at least $\mathscr{C}^5 = 0$ in the E_8 cases and $\mathscr{C}^3 = 0$ otherwise.

The BPS condition also requires the dilatino fields to be left invariant by some supersymmetry generators. In order for a Killing spinor to satisfy

$$\delta|\chi\rangle_\alpha = 0 \quad \Rightarrow \qquad e^\mu_a \sigma^a{}_\alpha{}^\beta \left(\epsilon^i_\beta\, a_i + \varepsilon_{\beta\gamma}\epsilon^\gamma_i a^i\right)|P_\mu\rangle = 0 \tag{2.44}$$

the charge state vector must satisfy

$$\left(\epsilon^i_\alpha a_i + \varepsilon_{\alpha\beta}\epsilon^\beta_i a^i\right)|\mathscr{C}\rangle = 0 \tag{2.45}$$

where $(\epsilon^i_\alpha, \epsilon^\alpha_i)$ is the asymptotic (for $r \to \infty$) value of the Killing spinor. As before, for $\mathcal{N} \leq 4$ the state $|\mathscr{C}\rangle$ may require an extra label, such that (2.45) gets replaced by

$$\left(\epsilon^i_\alpha a_i + \varepsilon_{\alpha\beta}\epsilon^\beta_i a^i\right)|\mathscr{C}, \mathcal{A}\rangle = 0 \tag{2.46}$$

The simple equation (2.45) (or 2.46)) is a key result of this paper: it encapsulates all the information about solutions of the equations of motion with residual supersymmetry and allows *a complete analysis of the BPS sector* (as we will see below, (2.45) is a stronger condition than the Killing spinor equation). Furthermore, we will show how the analysis of the BPS conditions can be reduced to simple calculations with fermionic oscillators by means of (2.45). Since $\mathscr{C}$ does not involve the angular momentum parameter a, we recover the (known) result that the BPS analysis is not sensitive to angular momentum. We note that (2.45) takes the form of a 'Dirac equation' for the $\mathfrak{H}^*$ spinor $|\mathscr{C}\rangle$, with the 'γ-matrices' (a_i, a^i) and the supersymmetry parameter (ϵ^i, ϵ_i) as the 'momentum'.

Multiplying equation (2.45) by its conjugate equation, and contracting the antichiral Weyl Spin$^*(2\mathcal{N})$ indices, one gets the integrability condition

$$\begin{pmatrix} \delta^\beta_\alpha \left(\langle\mathscr{C}|\mathscr{C}\rangle\, \delta^i_j + \frac{1}{2}\langle\mathscr{C}|[a^i, a_j]|\mathscr{C}\rangle\right) & \varepsilon_{\alpha\beta}\,\langle\mathscr{C}|a^i a^j|\mathscr{C}\rangle \\ -\varepsilon^{\alpha\beta}\,\langle\mathscr{C}|a_i a_j|\mathscr{C}\rangle & \delta^\alpha_\beta \left(\langle\mathscr{C}|\mathscr{C}\rangle\, \delta^j_i - \frac{1}{2}\langle\mathscr{C}|[a^j, a_i]|\mathscr{C}\rangle\right) \end{pmatrix} \begin{pmatrix} \epsilon^j_\beta \\ \epsilon^\beta_j \end{pmatrix} = 0 \tag{2.47}$$

necessary for a supersymmetry parameter to correspond to an unbroken supersymmetry generator. This equation decomposes into two inequivalent representations upon SO$^*(2\mathcal{N})$: first of all, we recover the condition $c^2 \equiv \langle\mathscr{C}|\mathscr{C}\rangle = 0$; secondly, we get

$$\mathcal{Z}(\epsilon) \equiv \begin{pmatrix} \frac{1}{2}\,\delta^\beta_\alpha\,\langle\mathscr{C}|[a^i, a_j]|\mathscr{C}\rangle & \varepsilon_{\alpha\beta}\,\langle\mathscr{C}|a^i a^j|\mathscr{C}\rangle \\ -\varepsilon^{\alpha\beta}\,\langle\mathscr{C}|a_i a_j|\mathscr{C}\rangle & -\frac{1}{2}\,\delta^\alpha_\beta\,\langle\mathscr{C}|[a^j, a_i]|\mathscr{C}\rangle \end{pmatrix} \begin{pmatrix} \epsilon^j_\beta \\ \epsilon^\beta_j \end{pmatrix} = 0\,. \tag{2.48}$$

[13] By 'Bogomolny matrix' we mean the matrix on the right hand side of the superalgebra when acting on the asymptotic free-particle states in four dimensions. This matrix is a function of the masses and central charges, and has vanishing determinant for BPS states.

The existence of unbroken supersymmetry generators thus requires both $c^2 = 0$ and that the matrix $\mathscr{Z}$, transforming under $\mathrm{SO}^*(2\mathcal{N})$ in the adjoint representation, leaves invariant the associated spinor parameter ϵ^i_α.

In the foregoing section we identified the *charge matrix* $\mathscr{C}$ by means of the conserved charges of the three-dimensional theory, whereas in this section we have been working with the *state* $|\mathscr{C}\rangle$, or a multiplet $|\mathscr{C},\mathcal{A}\rangle$ of such states. The two descriptions are obviously related, as the matrix $\mathscr{C}$ and state $|\mathscr{C}\rangle$ (or the multiplet $\{|\mathscr{C},\mathcal{A}\rangle\}$ for $\mathcal{N}\leq 4$) contain the same number of charge degrees of freedom, but writing down a general formula is neither easy nor really helpful because the most convenient conventions usually depend upon the properties of the specific groups $\mathfrak{G}$ and $\mathfrak{H}^*$. For $\mathcal{N}=5$ we will spell out the relation between the matrix $\mathscr{C}$ and the state vector $|\mathscr{C}\rangle$ explicitly in eq. (3.33) of section 3.3.

3 Solving the BPS conditions

Let us now proceed to analyse the BPS condition (2.45) case by case for various values of $\mathcal{N}$. As the conditions are the same even in the presence of several matter multiplets (for $\mathcal{N}\leq 4$), we will suppress the extra index $\mathcal{A}$ in this section. The state vector $|\mathscr{C}\rangle$ of charges has the following general form (cf. appendix B)

$$|\mathscr{C}\rangle \equiv \Big(W + Z_{ij}a^i a^j + \Sigma_{ijkl}a^i a^j a^k a^l + \cdots \Big)\,|0\rangle\ . \tag{3.1}$$

Here $W \equiv m + in$ is the complex gravitational charge (mass and NUT parameter), $Z_{ij}\equiv Q_{ij}+iP_{ij}$ are the electromagnetic charges, and Σ_{ijkl} are the scalar charges (which will turn out to depend on the other charges). Further charge components appear for $\mathcal{N}\geq 6$. For $\mathrm{Spin}^*(2\mathcal{N})$ the conjugate spinor is (see appendix B)[14]

$$\langle\mathscr{C}| = \langle 0|\Big(\bar{W} + Z^{ij}a_i a_j + \Sigma^{ijkl}a_i a_j a_k a_l + \cdots\Big) \tag{3.2}$$

from which we compute the norm, hence the BPS parameter, as[15]

$$c^2 = \langle\mathscr{C}|\mathscr{C}\rangle = |W|^2 - 2Z_{ij}Z^{ij} + 24\Sigma_{ijkl}\Sigma^{ijkl} - \cdots \tag{3.3}$$

3.1 General discussion and results for $\mathcal{N}\leq 5$

Equation (2.45) can now be decomposed with respect to $\oplus_p \bigwedge^{2p-1}\mathbb{C}^{\mathcal{N}}$, that is, the oscillator basis $a^i|0\rangle$, $a^i a^j a^k|0\rangle$,... The one-form component reads, for all $\mathcal{N}$,

$$2Z_{ij}\epsilon^j_\alpha - \varepsilon_{\alpha\beta}W\epsilon^\beta_i = 0 \tag{3.4}$$

The parameter W being non-zero for any non-trivial regular solution, we obtain that the spinor parameter associated to an unbroken supersymmetry generator satisfies

$$\epsilon^\alpha_i = -\frac{2}{W}\varepsilon^{\alpha\beta}Z_{ij}\epsilon^j_\beta \tag{3.5}$$

[14] Note the plus sign on all terms, which is related to the signature of the $\mathrm{Spin}^*(2\mathcal{N})$ scalar product.
[15] With a factor of $\frac{1}{2}$ in the case of maximal supergravity because the multiplet is self-dual.

relating the Killing spinor to its complex conjugate by a kind of symplectic Majorana condition. Hence, for all $\mathcal{N}$,

$$\frac{4}{|W|^2} Z^{ik} Z_{jk} \epsilon^j_\alpha = \epsilon^i_\alpha \ . \tag{3.6}$$

At this point it is advantageous to switch to a diagonal basis for the matrix Z_{ij}, which can be reached by conjugating with a suitable SU($\mathcal{N}$) matrix,

$$Z_{ij} \ \cong \ \frac{1}{2} \begin{pmatrix} 0 & z_1 & 0 & 0 & \cdots \\ -z_1 & 0 & 0 & 0 & \ddots \\ 0 & 0 & 0 & z_2 & \ddots \\ 0 & 0 & -z_2 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{3.7}$$

for $\mathcal{N} = 2M$, and

$$Z_{ij} \ \cong \ \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & z_1 & 0 & 0 & \ddots \\ 0 & -z_1 & 0 & 0 & 0 & \ddots \\ 0 & 0 & 0 & 0 & z_2 & \ddots \\ 0 & 0 & 0 & -z_2 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{3.8}$$

for $\mathcal{N} = 2M + 1$. Introducing M antisymmetric tensors ω^{m}_{ij} satisfying

$$\begin{aligned} \omega^{\mathrm{m}}_{ik}\, \omega^{\mathrm{n}\, jk} &= 0 \qquad \text{if} \quad \mathrm{m} \neq \mathrm{n} \\ \omega^{\mathrm{m}}_{ik}\, \omega^{\mathrm{m}\, jk} &= I^{\mathrm{m}}{}_i{}^j \end{aligned} \tag{3.9}$$

with the $I^{\mathrm{m}}{}_i{}^j$ being projectors onto the orthogonal 2-dimensional complex subspaces

$$\begin{aligned} I^{\mathrm{m}}{}_i{}^k I^{\mathrm{n}}{}_k{}^j &= 0 \qquad \text{if} \quad \mathrm{m} \neq \mathrm{n} \\ I^{\mathrm{m}}{}_i{}^k I^{\mathrm{m}}{}_k{}^j &= I^{\mathrm{m}}{}_i{}^j \\ I^{\mathrm{m}}{}_i{}^i &= 2 \, , \end{aligned} \tag{3.10}$$

we can re-express Z_{ij} in the form

$$Z_{ij} = \frac{1}{2} \sum_{\mathrm{m}} z_{\mathrm{m}} \ \omega^{\mathrm{m}}_{ij} \ . \tag{3.11}$$

Substituting this expression in equation (3.6) we obtain

$$\sum_{\mathrm{m}} \frac{|z_{\mathrm{m}}|^2}{|W|^2} I^{\mathrm{m}}{}_j{}^i \epsilon^j_\alpha = \epsilon^i_\alpha \ . \tag{3.12}$$

Consequently, the spinor parameter can have non-zero components *only in those subspaces for which* $|z_{\mathrm{m}}|^2 = |W|^2$. In accordance with established terminology *we shall speak of an*

$(n/\mathcal{N})$ *BPS solution if this relation is satisfied for n out of M values z_m.* For a spinor lying in the subspace m associated to the projector $I^m{}_j{}^i$ for which $|z_m|^2 = |W|^2$, equation (3.5) then becomes

$$\epsilon_i^\alpha = -\frac{z_m}{W}\varepsilon^{\alpha\beta}\omega^m_{ij}\epsilon^j_\beta \ . \tag{3.13}$$

Next, the 3-form component of equation (2.45) reads

$$4\Sigma_{ijkl}\epsilon^l_\alpha - \varepsilon_{\alpha\beta}Z_{[ij}\epsilon^\beta_{k]} = 0 \tag{3.14}$$

where Σ_{ijkl} are the scalar charges. Together with (3.5), this equation gives that

$$\Big(\Sigma_{ijkl} - \frac{1}{2W}Z_{[ij}Z_{kl]}\Big)\epsilon^l_\alpha = 0 \ . \tag{3.15}$$

This equation is again valid for all $\mathcal{N}$. It is trivially satisfied for $\mathcal{N} = 3$; for $\mathcal{N} = 4$ and $\mathcal{N} = 5$ it implies

$$\Sigma_{ijkl} = \frac{1}{2W}Z_{[ij}Z_{kl]} \ , \tag{3.16}$$

which is consistent with the 5-form component of equation (2.45)

$$\Sigma^{[ijkl}\epsilon^{m]}_\alpha = 0 \tag{3.17}$$

for $\mathcal{N} = 5$. For these values of $\mathcal{N}$, we have thus made completely explicit our previous claim that the scalar charges are not independent, but depend on the electromagnetic charges via eq. (3.16). As we will see latter, (3.16) is also the general solution of the equation $\mathscr{C}^3 = c^2\mathscr{C}$ for both $\mathcal{N} = 4$ and 5. Finally, we emphasise that, for $\mathcal{N} > 5$, the formula (3.16) is *not* valid in general, unless the BPS degree is sufficiently high.

3.2 $\mathcal{N} = 6$ and $\mathcal{N} = 8$ supergravity

We now proceed directly to $\mathcal{N} = 8$ because the case $\mathcal{N} = 6$ is most conveniently obtained by consistent truncation of $\mathcal{N} = 8$. For maximal supergravity, the scalar charge vector is given by

$$\begin{aligned}|\mathscr{C}\rangle = \Big(&W + Z_{ij}a^ia^j + \Sigma_{ijkl}a^ia^ja^ka^l + \frac{1}{6!}\varepsilon_{ijklmnpq}\,Z^{ij}\,a^ka^la^ma^na^pa^q \\ &+ \frac{1}{8!}\,\varepsilon_{ijklmnpq}\,\bar{W}\,a^ia^ja^ka^la^ma^na^pa^q\Big)|0\rangle \ .\end{aligned} \tag{3.18}$$

Its irreducibility as a Spin$^*(16)$ representation, that is, the condition $|\mathscr{C}\rangle = \star|\mathscr{C}\rangle$ (cf. appendix B) requires that the scalar charges are complex self-dual, viz.

$$\Sigma_{ijkl} = \frac{1}{4!}\varepsilon_{ijklmnpq}\Sigma^{mnpq} \ . \tag{3.19}$$

By self-duality the p-form component of equation (2.45) is equivalent to its $(\mathcal{N} - p)$-form component. The one-form and three-form components of this equation were already given in (3.5) and (3.15), respectively. However, unlike for $\mathcal{N} \leq 5$, we now no longer can 'peel off' the parameter ϵ^l_α from equation (3.15) in general, so formula (3.16) may fail.

For $\frac{1}{8}$ BPS solutions, we have $|z_1| = |W|$, whereas $|z_m| \neq |W|$ for $m = 2,3,4$. The non-vanishing components of (3.15) are then orthogonal, and they determine part of the scalar charges Σ_{ijkl}. The remaining components of Σ_{ijkl} can then be deduced from the self-duality constraint (3.19), in such a way that all scalar charges are determined as functions of the Z_{ij}, but (3.16) is not satisfied for all components as $W^{-1}Z_{[ij}Z_{kl]}$ need not be complex self-dual in general.

For $\frac{1}{4}$ BPS solutions, $|z_1| = |z_2| = |W|$ and $|z_3|, |z_4| \neq |W|$. In this case the components corresponding to the two different spinors overlap, although the formula (3.16) is still not valid for all components. The electromagnetic charges must satisfy constraints in order to be compatible with the self-duality constraint: inspection shows that (3.15) now implies

$$\frac{z_1 z_3}{W} = \frac{\bar{z}_2 \bar{z}_4}{\bar{W}} \qquad \frac{z_1 z_4}{W} = \frac{\bar{z}_2 \bar{z}_3}{\bar{W}} \,. \tag{3.20}$$

Therefore $|z_3|^2 = |z_4|^2$, and we conclude that *there cannot exist $\frac{3}{8}$ BPS asymptotically flat stationary solutions of $\mathcal{N} = 8$ supergravity* (which would require $|z_1| = |z_2| = |z_3| = |W| \neq |z_4|$). Finally, for $\frac{1}{2}$ BPS solutions, (3.15) is valid for any spinor parameter, and we at last recover (3.16). By self-duality, the electromagnetic charges must then satisfy

$$\frac{1}{2W} Z_{[ij} Z_{kl]} = \frac{1}{4!} \varepsilon_{ijklmnpq} \frac{1}{2\bar{W}} Z^{mn} Z^{pq} \,. \tag{3.21}$$

The formulas for $\mathcal{N} = 6$ can be obtained by truncation of the above results. However, $\mathcal{N} = 6$ supergravity is somewhat special because its bosonic sector, with the coset space $\mathrm{SO}^*(12)/\mathrm{U}(6)$, is identical to the bosonic sector of the magic $\mathcal{N} = 2$ supergravity [29]. The two theories differ only in their fermionic sectors, both of which can be obtained by truncation of $\mathcal{N} = 8$ supergravity. While the bosons are truncated in the same way to give the coset $\mathrm{SO}^*(12)/\mathrm{U}(6)$ for both the $\mathcal{N} = 2$ and $\mathcal{N} = 6$ cases, one retains six gravitinos and 26 spin-$\frac{1}{2}$ fermions in the $\mathcal{N} = 6$ theory, whereas for the $\mathcal{N} = 2$ theory one retains the complementary set of two gravitinos and 30 spin-$\frac{1}{2}$ fermions (the latter belong to 15 vector multiplets coupled to the $\mathcal{N} = 2$ graviton multiplet), such that there are altogether 32 fermionic degrees of freedom in each case. In other words, the bosonic sector by itself 'does not know' whether it belongs to $\mathcal{N} = 6$ supergravity or to the magic $\mathcal{N} = 2$ theory.

These features can be seen directly from the form of the truncated charge vector which is represented by the state

$$\begin{aligned} |\mathscr{C}\rangle = &\Big(W + \bar{Z} a^7 a^8\Big)|0\rangle + \big(Z_{ij} + \Sigma_{ij}\, a^7 a^8\big)\, a^i a^j |0\rangle + \frac{1}{4!}\varepsilon_{ijklmn}\Big(\Sigma^{ij} + Z^{ij}\, a^7 a^8\Big) a^k a^l a^m a^n |0\rangle \\ &+ \frac{1}{6!}\varepsilon_{ijklmn}\,\big(Z + \bar{W}\, a^7 a^8\big)\, a^i a^j a^k a^l a^m a^n |0\rangle \end{aligned} \tag{3.22}$$

and which can be directly obtained from (3.18) by truncation. Here $i, j, \dots = 1, \dots, 6$ label the $\mathcal{N} = 6$ oscillators while a^7 and a^8 correspond to the supercharges of the $\mathcal{N} = 2$ theory. When viewed as an $\mathcal{N} = 2$ theory, equation (2.45) reduces to its 1-form component which decomposes into (3.5) for the spinors ϵ^7_α and ϵ^8_α and a matter component

$$\Sigma_{ij}\, \epsilon^8_\alpha - \varepsilon_{\alpha\beta}\, Z_{ij}\, \epsilon^\beta_7 = 0 \tag{3.23}$$

which immediately yields the $\mathcal{N} = 2$ $\frac{1}{2}$ BPS conditions, $|Z| = |W|$ and

$$\Sigma_{ij} = \frac{1}{W} \bar{Z} Z_{ij} \ . \tag{3.24}$$

For $\mathcal{N} = 6$, on the other hand, we get the 3-form and the 5-form equations

$$\epsilon_\alpha^{[k} \Big(\Sigma^{ij]} - \frac{1}{4W} \varepsilon^{ij]mnpq} Z_{mn} Z_{pq} \Big) = 0 \qquad \Big(\Sigma_{ij} - \frac{1}{W} \bar{Z}\, Z_{ij} \Big) \epsilon_\alpha^j = 0 \tag{3.25}$$

where we have already substituted the solution (3.5) for the supersymmetry generator ϵ_α^i. For $\frac{1}{6}$ BPS solutions, the non-trivial components of these equations are orthogonal, and again suffice to determine the scalar charges Σ^{ij} as functions of the others. For more supersymmetric solutions, the scalar charges are determined by equation (3.16) to be

$$\Sigma^{ij} = \frac{1}{4W} \varepsilon^{ijmnpq} Z_{mn} Z_{pq} \ . \tag{3.26}$$

Requiring consistency with (3.24) along the $\mathbb{C}^4$ subspace associated to the unbroken supersymmetries gives

$$\frac{Z \bar{z}_1}{\bar{W}} = \frac{z_2 z_3}{W} \qquad \frac{Z \bar{z}_2}{\bar{W}} = \frac{z_1 z_3}{W} \tag{3.27}$$

which is just the condition (3.20) in disguise. Because $|z_1|^2 = |z_2|^2 = |W|^2$, both equations reduce to

$$Z = \frac{z_1 z_2 z_3}{W^2} \ . \tag{3.28}$$

The charge Z is thus determined to be

$$Z = \frac{1}{6W^2} \varepsilon^{ijklmn} Z_{ij} Z_{kl} Z_{mn} \tag{3.29}$$

with $|Z|^2 = |z_3|^2$ for $\frac{1}{3}$ BPS solutions. For $\frac{1}{2}$ BPS solutions all the components of $\Sigma_{ij} - (\bar{Z}\, Z_{ij})/W$ must cancel and we get

$$\frac{1}{W} \bar{Z}\, Z_{ij} = \frac{1}{4\bar{W}} \varepsilon_{ijmnpq} Z^{mn} Z^{pq} \ . \tag{3.30}$$

eq.ivalently, the condition for a solution to preserve some supersymmetry in both the $\mathcal{N} = 2$ and the $\mathcal{N} = 6$ theories requires the remaining eigenvalues of Z_{ij} to be equal in modulus, which is consistent with the non-existence of $\frac{3}{8}$ BPS solutions in $\mathcal{N} = 8$ supergravity.

We conclude this subsection with a few comments on black hole entropy in $\mathcal{N} = 8$ supergravity. In that case, the constraints on the electromagnetic charges are related to extremality properties of the $E_{7(7)}$ invariant expression of the entropy [33]. For static solutions with $W = \bar{W} = m$ satisfying the $\frac{1}{4}$ BPS bound condition $|z_1| = |z_2| = m$ (and $|z_3|, |z_4|$ possibly different from m), equation (3.20) is strictly equivalent to the vanishing of the $E_{7(7)}$ invariant expression of the horizon area $A = 4\pi\sqrt{\Diamond(Z)}$, where

$$\begin{aligned} \Diamond(Z) \equiv Z_{ij} Z^{jk} Z_{kl} Z^{li} - \frac{1}{4} Z_{ij} Z^{ij} Z_{kl} Z^{kl} \\ + \frac{1}{96} \varepsilon_{ijklmnpq} Z^{ij} Z^{kl} Z^{mn} Z^{pq} + \frac{1}{96} \varepsilon^{ijklmnpq} Z_{ij} Z_{kl} Z_{mn} Z_{pq} \ . \end{aligned} \tag{3.31}$$

This proves the conjecture of [14] proposing the vanishing of the $E_{7(7)}$ invariant expression of the horizon area for $\frac{1}{4}$ BPS and $\frac{1}{2}$ BPS black holes. For asymptotically Taub–NUT solutions, W is complex, and the $\frac{1}{4}$ BPS condition (3.20) requires that the Ehlers U(1) invariant $\diamondsuit(W^{-\frac{1}{2}}Z)$ vanish. This leads us to conjecture that the expression for the horizon area of asymptotically Taub–NUT BPS black holes in maximal supergravity is

$$A = 4\pi|W|\sqrt{\diamondsuit(W^{-\frac{1}{2}}Z)}\ . \tag{3.32}$$

As a matter of fact, this expression is not in general invariant with respect to the standard action of $E_{7(7)}$ on the electromagnetic charges. This is not in contradiction with the U-duality invariance of the entropy, however, since the latter cannot be identified with the horizon area for asymptotically Taub–NUT spacetimes.

3.3 Relation to pure spinors

There is an intriguing link between the cubic constraint $\mathscr{C}^3 = c^2\mathscr{C}$ on the charge matrix and pure spinors in pure supergravity theories. Let us start with $\mathcal{N} = 5$ supergravity, for which the corresponding pure spinor equation is more familiar to physicists thanks to the work of N. Berkovits in superstring theory. The duality group of the three-dimensional theory is $E_{6(-14)}$ which admits a complex 27-dimensional faithful representation. With respect to the maximal subgroup $\mathrm{U}(1) \times \mathrm{Spin}^*(10)$, the **27** decomposes into $\mathbf{1} \oplus \mathbf{16} \oplus \mathbf{10}$ where **16** is the complex chiral spinor representation of $\mathrm{Spin}^*(10)$ and **10** the pseudo-real vector representation of $\mathrm{SO}^*(10)$. The charge matrix $\mathscr{C}$ can be defined in terms of the chiral spinor $|\mathscr{C}\rangle$ as

$$\mathscr{C} \equiv \begin{pmatrix} 0 & \langle\mathscr{C}| & 0 & 0 \\ |\mathscr{C}\rangle & 0 & a_j|\mathscr{C}^\star\rangle & a^j|\mathscr{C}^\star\rangle \\ 0 & \langle\mathscr{C}^\star|a^i & 0 & 0 \\ 0 & \langle\mathscr{C}^\star|a_i & 0 & 0 \end{pmatrix} \tag{3.33}$$

which is understood to act on a complex 27-dimensional vector $(\eta, |S\rangle, V^j, V_j)$. $|\mathscr{C}^\star\rangle$ is the antichiral spinor defined from the anti-involution $\star$

$$|\mathscr{C}^\star\rangle \equiv \star|\mathscr{C}\rangle = \varepsilon_{ijklm}\Big(\Sigma^{jklm}a^i + \frac{1}{3!}Z^{lm}\, a^i a^j a^k + \frac{1}{5!}\bar{W}\, a^i a^j a^k a^l a^m\Big)|0\rangle\ . \tag{3.34}$$

The formula (3.33) makes the claimed relation between the matrix $\mathscr{C}$ and the state vector $|\mathscr{C}\rangle$ completely explicit for $\mathcal{N} = 5$. Making use of the properties

$$\begin{gathered}\langle\mathscr{C}|\mathscr{C}\rangle = \langle\mathscr{C}^\star|\mathscr{C}^\star\rangle \\ \langle\mathscr{C}|a_i a_j|\mathscr{C}\rangle = -\langle\mathscr{C}^\star|a_i a_j|\mathscr{C}^\star\rangle \qquad \langle\mathscr{C}|a^i a_j|\mathscr{C}\rangle = \langle\mathscr{C}^\star|a_j a^i|\mathscr{C}^\star\rangle\,,\end{gathered} \tag{3.35}$$

the Fierz identity

$$a_i|\mathscr{C}^\star\rangle\langle\mathscr{C}|a^i + a^i|\mathscr{C}^\star\rangle\langle\mathscr{C}|a_i = -\frac{1}{2}\langle\mathscr{C}|a^i|\mathscr{C}^\star\rangle a_i - \frac{1}{2}\langle\mathscr{C}|a_i|\mathscr{C}^\star\rangle a^i \tag{3.36}$$

and its conjugate, we compute $\operatorname{Tr}\mathscr{C}^2 = 12\,\langle\mathscr{C}|\mathscr{C}\rangle$ and

$$\begin{aligned}\mathscr{C}^3 - c^2\mathscr{C} = \langle\mathscr{C}|a_k|\mathscr{C}^\star\rangle &\begin{pmatrix} 0 & \langle\mathscr{C}^\star|a^k & 0 & 0 \\ a^k|\mathscr{C}^\star\rangle & 0 & \big(\delta^k_j + \tfrac{1}{2}a^k a_j\big)|\mathscr{C}\rangle & \tfrac{1}{2}a^k a^j|\mathscr{C}\rangle \\ 0 & \langle\mathscr{C}|\tfrac{1}{2}a^i a^k & 0 & 0 \\ 0 & \langle\mathscr{C}|\big(\delta^k_i + \tfrac{1}{2}a_i a^k\big) & 0 & 0 \end{pmatrix} \\ + \langle\mathscr{C}|a^k|\mathscr{C}^\star\rangle &\begin{pmatrix} 0 & \langle\mathscr{C}^\star|a_k & 0 & 0 \\ a_k|\mathscr{C}^\star\rangle & 0 & \tfrac{1}{2}a_k a_j|\mathscr{C}\rangle & \big(\delta^j_k + \tfrac{1}{2}a_k a^j\big)|\mathscr{C}\rangle \\ 0 & \langle\mathscr{C}|\big(\delta^i_k + \tfrac{1}{2}a^i a_k\big) & 0 & 0 \\ 0 & \langle\mathscr{C}|\tfrac{1}{2}a_i a_k & 0 & 0 \end{pmatrix} \ . \end{aligned} \tag{3.37}$$

It follows that the constraint $\mathscr{C}^3 = c^2\mathscr{C}$ is strictly equivalent to the $\mathrm{Spin}^*(10)$ pure spinor constraint

$$\langle\mathscr{C}|\, a^i|\mathscr{C}^\star\rangle = 0 \qquad \langle\mathscr{C}|\, a_i|\mathscr{C}^\star\rangle = 0\ . \tag{3.38}$$

Here, we define a $\mathrm{Spin}^*(2\mathcal{N})$ pure spinor by the direct generalisation of the Cartan definition, that is by the requirement that $|\mathscr{C}^\star\rangle\langle\mathscr{C}|$ lies in the rank $\mathcal{N}$ antisymmetric tensor representation of $\mathrm{SO}^*(2\mathcal{N})$. The same computation in $\mathcal{N}=4$ pure supergravity shows that the cubic constraint (2.22) is strictly equivalent to the $\mathrm{Spin}^*(8)$ pure spinor constraint

$$\langle\mathscr{C}|\mathscr{C}^\star\rangle = 0 \tag{3.39}$$

where

$$|\mathscr{C}^\star\rangle \equiv \star\,|\mathscr{C}\rangle = \Big(\varepsilon_{ijkl}\,\Sigma^{ijkl} + \frac{1}{2}\varepsilon_{ijkl}\,Z^{ij}\,a^k a^l + \frac{1}{4!}\varepsilon_{ijkl}\,\bar{W}\,a^i a^j a^k a^l\Big)\,|0\rangle\ . \tag{3.40}$$

For practical computation it is much easier to consider the coset $\mathrm{Spin}(2,8)/(\mathrm{Spin}(2,6)\times \mathrm{U}(1))$ exploiting the isomorphism $\mathrm{Spin}(2,6)\cong\mathrm{Spin}^*(8)$. We postpone the proof of equivalence to the pure spinor constraint to section 6. For $\mathcal{N}=2$ and $\mathcal{N}=3$, there are no scalar charges and the equation $\mathscr{C}^3 = c^2\mathscr{C}$ is trivially satisfied by any element of the coset $\mathfrak{g}\ominus\mathfrak{h}^*$. This is in agreement with the fact that any $\mathrm{Spin}^*(4)$ or $\mathrm{Spin}^*(6)$ chiral spinor is pure.

The general solution of the $\mathrm{Spin}^*(2\mathcal{N})$ pure spinor constraint is

$$|\mathscr{C}\rangle = W\,\exp\left(\frac{1}{W}Z_{ij}a^i a^j\right)|0\rangle\ . \tag{3.41}$$

It is well defined only if $W\neq 0$ but, since $W = m+in$, it is natural to make this requirement. To prove (3.41), we use the fact that for a spinor satisfying $\langle\mathscr{C}|\mathscr{C}\rangle > 0$, there exists a $\mathrm{U}(1)\times\mathrm{Spin}^*(2\mathcal{N})$ transformation that rotates both the electromagnetic charges and the NUT charge to zero, such that in the new 'frame'

$$|\mathscr{C}\rangle = c|0\rangle\ . \tag{3.42}$$

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Then, from the definition of the anti-involution $\star$ (cf. appendix B), we have

$$|\mathscr{C}^\star\rangle\langle\mathscr{C}| = c^2 \star |0\rangle\langle 0| = \frac{c^2}{\mathcal{N}!}\,\varepsilon_{i_1\cdots i_{\mathcal{N}}}\,a^{i_1}\cdots a^{i_{\mathcal{N}}}|0\rangle\langle 0| = \frac{c^2}{\mathcal{N}!}\,\varepsilon_{i_1\cdots i_{\mathcal{N}}}\,a^{i_1}\cdots a^{i_{\mathcal{N}}} \tag{3.43}$$

where we have made use of the fact that we can replace $|0\rangle\langle 0|$ by the unit operator in this expression because the left state is fully occupied. To complete the proof, we only need to rotate the spinor back to its original frame (3.41); it is then easy to see that the above result gets replaced by a combination of products of $\mathcal{N}$ fermionic (creation *and* annihilation) oscillators corresponding to the $\mathcal{N}$-form representation of $\mathrm{Spin}^*(2\mathcal{N})$. Consequently, $|\mathscr{C}\rangle$ is a pure spinor for all $\mathcal{N}$.

Writing out the pure spinor condition for $\mathcal{N}=4$ and $\mathcal{N}=5$, we can easily see that it is equivalent to the equation

$$W\Sigma_{ijkl} = \frac{1}{2}Z_{[ij}Z_{kl]} \tag{3.44}$$

which coincides with the equation derived from the requirement for the solution to be BPS, cf. (3.16). In the preceding section, this condition and the BPS bound condition on the eigenvalues of the electromagnetic charges were enough for the solution to be BPS. We are now going to see that the orders of the zeros of the BPS parameter are indeed governed by the number of eigenvalues of the electromagnetic charges which satisfy the BPS bound. Inserting the solution (3.44) into the definition of the BPS parameter (3.3), we get

$$c^2 = |W|^2 - 2Z_{ij}Z^{ij} + \frac{2}{|W|^2}\Big(\big(Z_{ij}Z^{ij}\big)^2 - 2Z_{ij}Z^{jk}Z_{kl}Z^{li}\Big) \tag{3.45}$$

which reduces to

$$c^2 = \frac{\big(|W|^2-|z_1|^2\big)\big(|W|^2-|z_2|^2\big)}{|W|^2} \tag{3.46}$$

in terms of the eigenvalues z_1 and z_2 (the formula is also valid for $\mathcal{N}=2,3$ with $z_2=0$). Without NUT charge ($|W|^2=m^2$), c^2 to a given power is thus proportional to the determinant of the Bogomolny matrix obtained from the four-dimensional supersymmetry algebra projected on an asymptotically free massive particle state. As we just discussed, once the constraint (2.22) is solved, the number of preserved supersymmetries can be derived from this determinant. It follows also from equation (3.46) that all the extremal solutions admitting a nilpotent charge matrix $\mathscr{C}$ are BPS, and thus the moduli space of stationary black holes is given by the union of the $\mathrm{U}(1)\times\mathrm{Spin}^*(2\mathcal{N})$-orbits of non-extremal Kerr–Taub–NUT black holes and the orbits of BPS black holes.

For $\mathcal{N}=6$, the $E_{7(-5)}$ constraint $\mathscr{C}^3=c^2\mathscr{C}$ is equivalent to the $\mathrm{SL}(2,\mathbb{R})\times\mathrm{Spin}^*(12)$ invariant equation

$$\begin{aligned}\langle\mathscr{C}|a^ia_j|\mathscr{C}\rangle\,a^j|\mathscr{C}\rangle + \langle\mathscr{C}|a^ia^j|\mathscr{C}\rangle\,a_j|\mathscr{C}\rangle - \langle\mathscr{C}^\star|a^ia_j|\mathscr{C}\rangle\,a^j|\mathscr{C}^\star\rangle - \langle\mathscr{C}^\star|a^ia^j|\mathscr{C}\rangle\,a_j|\mathscr{C}^\star\rangle &= 0\\ \langle\mathscr{C}^\star|a^ia_j|\mathscr{C}^\star\rangle\,a^j|\mathscr{C}^\star\rangle + \langle\mathscr{C}^\star|a^ia^j|\mathscr{C}^\star\rangle\,a_j|\mathscr{C}^\star\rangle - \langle\mathscr{C}|a^ia_j|\mathscr{C}^\star\rangle\,a^j|\mathscr{C}\rangle - \langle\mathscr{C}|a^ia^j|\mathscr{C}^\star\rangle\,a_j|\mathscr{C}\rangle &= 0\end{aligned} \tag{3.47}$$

In this case, this equation does not reduce any more to a quadratic constraint on the spinor $|\mathscr{C}\rangle$. For $c^2\neq 0$, the scalar charge Σ^{ij} is generally a non-rational function of W,

Z_{ij} and Z. For instance, the solution of $\mathscr{C}^3 = c^2\mathscr{C}$ for electromagnetic charges that are very small compared to the parameter W defines Σ^{ij} as an infinite formal series in powers of $\frac{Z_{ij}}{W}$, $\frac{\bar{Z}}{W}$ and their complex conjugates, and the resulting expression cannot in general be written in closed form. The BPS parameter thus is not simply proportional to the product of the determinants of the Bogomolny matrices of the $\mathcal{N} = 2$ and $\mathcal{N} = 6$ supergravities associated to this bosonic theory. Nevertheless, the Spin*(12) pure spinors define solutions of equation (3.47), although not all its solutions define pure spinors. The Spin*(12) pure spinor condition reads

$$\frac{1}{2}\langle\mathscr{C}|[a^i, a_j]|\mathscr{C}^\star\rangle = 0 \qquad \langle\mathscr{C}|a^i a^j|\mathscr{C}^\star\rangle = 0 \qquad \langle\mathscr{C}|a_i a_j|\mathscr{C}^\star\rangle = 0\ . \tag{3.48}$$

Note that although these equations are invariant under the action of Spin*(12), they are not invariant under the action of SL(2, $\mathbb{R}$) in general, and so the general solution of $\mathscr{C}^3 = c^2\mathscr{C}$ cannot be a pure spinor. The pure spinor condition in components reads

$$8\,\Sigma^{ik}Z_{jk} = \delta^i_j\big(2Z_{ij}\Sigma^{ij} - WZ\big) \qquad W\Sigma^{ij} = \frac{1}{4}\varepsilon^{ijklmn}Z_{kl}Z_{mn} \qquad ZZ_{ij} = \frac{1}{4}\varepsilon_{ijklmn}\Sigma^{kl}\Sigma^{mn}\ . \tag{3.49}$$

The general solution determines both the scalar charge Σ^{ij} and the electromagnetic charge Z to be

$$\Sigma^{ij} = \frac{1}{4W}\varepsilon^{ijklmn}Z_{kl}Z_{mn} \qquad Z = \frac{1}{6W^2}\varepsilon^{ijklmn}Z_{ij}Z_{kl}Z_{mn}\ . \tag{3.50}$$

Note that according to equation (3.26) and (3.29), the $\frac{1}{3}$ and the $\frac{1}{2}$ BPS solutions of $\mathcal{N} = 6$ supergravity do satisfy these equations, and the charge matrix associated to such a solution defines a pure spinor. In general, for a charge matrix satisfying the pure spinor equation, one recovers the property that the BPS parameter is proportional to the determinant of the Bogomolny matrix, viz.

$$c^2 = \frac{\big(|W|^2 - |z_1|^2\big)\big(|W|^2 - |z_2|^2\big)\big(|W|^2 - |z_3|^2\big)}{|W|^4}\ . \tag{3.51}$$

Such a restricted solution is $\frac{1}{2}$ BPS in the quaternionic $\mathcal{N} = 2$ magic supergravity if and only if it is $\frac{1}{2}$ BPS in $\mathcal{N} = 6$ supergravity. Although the general solution of (3.47) is generically not a pure spinor, it follows from the transitivity property of SL(2, $\mathbb{R}$) $\times$ Spin*(12) on the moduli space of non-extremal black holes that $\mathscr{C}$ is in the SL(2, $\mathbb{R}$)-orbit of a pure spinor for $c > 0$. The general solution of (3.47) can thus be parametrised as follows

$$\begin{aligned}
W &= \cosh u\, X + \sinh u\, e^{i\alpha}\,\frac{1}{6\bar{X}^2}\varepsilon_{ijklmn}X^{ij}X^{kl}X^{mn}\\
Z &= \cosh u\,\frac{1}{6X^2}\varepsilon^{ijklmn}X_{ij}X_{kl}X_{mn} + \sinh u\, e^{i\alpha}\,\bar{X}\\
Z_{ij} &= \cosh u\, X_{ij} + \sinh u\, e^{i\alpha}\,\frac{1}{4\bar{X}}\varepsilon_{ijklmn}X^{kl}X^{mn}\\
\Sigma^{ij} &= \cosh u\,\frac{1}{4X}\varepsilon^{ijklmn}X_{kl}X_{mn} + \sinh u\, e^{i\alpha}\,X^{ij}\ .
\end{aligned} \tag{3.52}$$

By taking appropriate limits, one obtains extremal solutions which are BPS either in $\mathcal{N} = 6$ supergravity or in the corresponding magic $\mathcal{N} = 2$ supergravity associated to the quaternions. Nevertheless, this does not prove that there are no non-BPS extremal solutions with $c = 0$.

Although we have not written out explicitly the quintic equation (2.21) for maximal supergravity, the requirement of Spin$^*(16)$ covariance completely fixes the expression for the scalar charges in terms of the other charges when $|Z_{ij}| \ll |W|$. As for $\mathcal{N} = 6$, the expression for the scalar charges can be expanded into an infinite series in powers of (Z_{ij}/W) in such a way that the solution of (2.21) defines the scalar charges as non-rational functions of the others. At low orders, we have[16]

$$
\begin{aligned}
\Sigma_{ijkl} = & \Big(1 + \frac{1}{24 w^3 \bar{w}} \varepsilon^{mnpqrstu} Z_{mn} Z_{pq} Z_{rs} Z_{tu} + \frac{1}{24 w \bar{w}^3} \varepsilon_{mnpqrstu} Z^{mn} Z^{pq} Z^{rs} Z^{tu}\Big) \\
& \cdot \Big(\frac{1}{2w} Z_{[ij} Z_{kl]} + \frac{1}{48\bar{w}} \varepsilon_{ijklvwxy} Z^{vw} Z^{xy}\Big) \\
& - \frac{5}{w^2 \bar{w}} Z_{[ij} Z_{kl} Z_{mn]} \Big(Z^{mn} - \frac{6}{|w|^2} Z_{pq} Z^{[mn} Z^{pq]}\Big) \\
& - \frac{5}{24 w \bar{w}^2} \varepsilon_{ijklmnpq} Z^{[mn} Z^{pq} Z^{rs]} \Big(Z_{rs} - \frac{6}{|w|^2} Z^{tu} Z_{[rs} Z_{tu]}\Big) + \mathcal{O}\left(\frac{Z^8}{w^7}\right) . \qquad (3.53)
\end{aligned}
$$

It follows that the BPS parameter does not reduce to an expression proportional to the determinant of the Bogomolny matrix for asymptotically Minkowski solutions.

The charge matrix transforms as a Majorana–Weyl spinor of Spin$^*(16)$, whereas the pure spinor equation is defined for complex spinors. The pure spinor equation for a Majorana–Weyl spinor implies that $\langle \mathscr{C} | \mathscr{C} \rangle = 0$, and so there is no non-trivial solution in an Euclidean case with the group Spin$(2\mathcal{N})$. However, since the scalar product $\langle \mathscr{C} | \mathscr{C} \rangle$ is indefinite for Spin$^*(16)$, there do exist non-trivial solutions in this case. Indeed, if one writes down the constraints

$$
|\mathscr{C}\rangle = w \ \exp\left(\frac{1}{W} Z_{ij} a^i a^j\right) |0\rangle = \star \, w \ \exp\left(\frac{1}{W} Z_{ij} a^i a^j\right) |0\rangle \ , \qquad (3.54)
$$

one gets exactly the $\mathcal{N} = 8$ constraints necessary for the corresponding solution to be $\frac{1}{2}$ BPS. As a result, the moduli space of $\frac{1}{2}$ BPS asymptotically flat stationary single-particle solutions of $\mathcal{N} = 8$ supergravity is isomorphic to the space of Spin$^*(16)$ Majorana–Weyl pure spinors.

4 Isotropy subgroups of BPS solutions

The formalism developed in the previous sections affords a convenient tool to investigate, and in fact completely characterise, all the BPS orbits for different $\mathcal{N}$, thus furnishing a proof for a number of conjectures that have been made in the literature.

4.1 Pure supergravities for $\mathcal{N} \leq 5$

From the results of the previous section, it follows that the moduli space of solutions of the cubic equation (2.22) is strictly equivalent to the space of pure spinors of Spin$^*(2\mathcal{N})$

[16] Note that the value of $|Z_{ij}/W|$ is larger than the radius of convergence of the formal series for BPS solutions.

for all $\mathcal{N} \leq 5$. Defining $\Omega_{ij} \equiv (2/\mathrm{W})Z_{ij}$, equation (3.41) tells us that the general solution can be written as

$$|\mathscr{C}\rangle = \mathrm{W}\ \exp\left(\frac{1}{2}\Omega_{ij}a^i a^j\right)|0\rangle\ . \tag{4.1}$$

We emphasise again that for $\mathcal{N} \leq 5$ this form of $|\mathscr{C}\rangle$ is valid also for non-BPS solutions: in that case we simply set $\Omega_{ij} = 0$ because we can use the duality group to rotate the solution to a 'frame' where it is a pure Kerr–Taub–NUT solution with (complex) parameter W. We also recall that for $\mathcal{N} \geq 3$, the group $\mathrm{Spin}^*(2\mathcal{N})$ is always accompanied by an extra U(1) which must be taken into account when analysing the residual symmetries.

The action of $\mathfrak{u}(1) \oplus \mathfrak{spin}^*(2\mathcal{N})$ on the above spinor can be worked out by means of the formulas given in appendix B to give[17]

$$\delta\,|\mathscr{C}\rangle = \frac{1}{2}\Big(\big(2\,\Lambda_i{}^k\Omega_{kj} + \Lambda_{ij} + \Omega_{ik}\,\Lambda^{kl}\Omega_{lj}\big)a^i a^j + \Omega_{ij}\Lambda^{ij} - \Lambda_i{}^i - i\lambda\Big)\,|\mathscr{C}\rangle\ , \tag{4.2}$$

where λ parametrises the $\mathfrak{u}(1)$ transformation. For a matrix charge $\mathscr{C}$ corresponding to a $\frac{n}{\mathcal{N}}$ BPS solution, the matrix Ω_{ij} can be moved via a $\mathrm{Spin}^*(2\mathcal{N})$ rotation to a symplectic form on a subspace $\mathbb{C}^{2n} \subset \mathbb{C}^{\mathcal{N}}$. In order to analyse the isotropy subgroup of $\mathrm{U}(1) \times \mathrm{Spin}^*(2\mathcal{N})$ corresponding to such a spinor, it is convenient to decompose the $\mathrm{U}(\mathcal{N})$ indices according to the product $\mathrm{U}(2n) \times \mathrm{U}(\mathcal{N} - 2n)$ into unbarred ones $A, B, \dots = 1, \dots, 2n$ and barred ones $\bar{A}, \bar{B}, \dots = 1, \dots, \mathcal{N} - 2n$, respectively. Splitting the equations (4.2) in this way and demanding $\delta|\mathscr{C}\rangle = 0$, we arrive at

$$\begin{aligned} 2\Lambda_{[A}{}^C\Omega_{C|B]} + \Lambda_{AB} + \Omega_{AC}\Lambda^{CD}\Omega_{DB} &= 0 & \qquad \Lambda_{\bar{A}}{}^C\Omega_{CB} + \Lambda_{\bar{A}B} &= 0 \\ -i\lambda + \Omega_{AB}\Lambda^{AB} - \Lambda_A{}^A - \Lambda_{\bar{A}}{}^{\bar{A}} &= 0 & \Lambda_{\bar{A}\bar{B}} &= 0\ . \end{aligned} \tag{4.3}$$

Taking the symplectic trace of the first equation (with $\Omega_{AC}\Omega^{CB} = -\delta_A^B$), we get

$$2\Lambda_A{}^A = \Omega^{AB}\Lambda_{AB} + \Omega_{AB}\Lambda^{AB}\ . \tag{4.4}$$

Let us first consider the subgroup of the isotropy group lying in the maximal compact subgroup $\mathrm{U}(1)\times\mathrm{U}(\mathcal{N}) \subset \mathrm{U}(1)\times\mathrm{Spin}^*(2\mathcal{N})$. In this case the constraints on the Lie algebra generators imply $\Lambda_{\bar{A}}{}^B = 0$ and $\Lambda_{[A}{}^C\Omega_{C|B]} = 0$, whence the generators inside $\mathrm{U}(2n)$ must leave invariant the symplectic form Ω_{AB}, and therefore generate the subgroup $\mathrm{Sp}(n) \equiv U\mathrm{Sp}(2n) \subset \mathrm{U}(2n)$. From the third equation in (4.3), we deduce that λ is determined in terms of the other parameters, hence is not independent. The maximal compact subgroup of the isotropy subgroup is thus $\mathrm{Sp}(n) \times \mathrm{U}(\mathcal{N} - 2n)$.

To analyse the non-compact generators we define

$$\Lambda^{\pm}_{AB} := \frac{1}{2}\Big(\Lambda_{AB} \pm \Omega_{AC}\Omega_{BD}\Lambda^{\pm\,CD}\Big) \quad \Rightarrow \qquad \Lambda^{\pm}_{AB} = \pm\Omega_{AC}\Omega_{BD}\Lambda^{\pm\,CD}\ . \tag{4.5}$$

Then we see that Λ^{+}_{AB} drops out from the first equation in (4.3), but there is nevertheless still one constraint on it. Namely, from (4.5) we get $\Omega^{AB}\Lambda^{\pm}_{AB} = \pm\Omega_{AB}\Lambda^{\pm\,AB}$; thus,

[17] Recall that raising or lowering indices on Λ corresponds to complex conjugation.

$\Omega^{AB}\Lambda^+_{AB}$ is real, while $\Omega^{AB}\Lambda^-_{AB}$ is imaginary. From the third equation in (4.3) we then deduce that

$$\Omega^{AB}\Lambda^+_{AB} = 0 \tag{4.6}$$

(all other terms being pure imaginary). Together with Sp(n) these parameters combine to give the *non-compact* real form $\mathrm{SU}^*(2n) \subset \mathrm{SL}(2n,\mathbb{C})$.[18] In terms of fermionic oscillators, a given element of $\mathrm{SU}^*(2n)$ is defined via the following generators of $\mathrm{SL}(2n,\mathbb{C})$

$$\begin{aligned}
\mathbf{X}^A{}_B &\equiv \frac{1}{2}\Big(a^A a_B - \Omega^{AC}\Omega_{BD}\, a^D a_C\Big)\\
\mathbf{X}^{AB} &\equiv \frac{1}{2}\Big(a^A a^B - \Omega^{AC}\Omega^{BD} a_C a_D\Big) - \frac{1}{4n}\Omega^{AB}\Big(\Omega_{CD} a^C a^D - \Omega^{CD} a_C a_D\Big)
\end{aligned} \tag{4.7}$$

with an anti-Hermitean matrix $\Lambda_A{}^B = -\Lambda^B{}_A$ satisfying $\Lambda_A{}^B = -\Omega_{AC}\Omega^{BD}\Lambda_D{}^B$, and a traceless element Λ^+_{AB} as

$$\mathbf{X}(\Lambda) = \Lambda_A{}^B\mathbf{X}^A{}_B + \frac{1}{2}(\Lambda^+_{AB}\mathbf{X}^{AB} + \Lambda^{+\,AB}\mathbf{X}_{AB}) \tag{4.8}$$

where $\mathbf{X}_{AB} \equiv (\mathbf{X}^{AB})^\dagger$. Although the fundamental representation of SU($2n$) is complex for $n > 1$, the fundamental representation of $\mathrm{SU}^*(2n)$ is pseudo-real. Indeed, in order to be consistent with supersymmetry, the action of $\mathrm{SU}^*(2n)$ on a Killing spinor ϵ^A_α must preserve the reality condition (3.5), i.e. $\epsilon^A_\alpha = -\Omega^{AB}\varepsilon_{\alpha\beta}\epsilon^\beta_B$:

$$\big[\,\mathbf{X}(\Lambda)\,,\;\epsilon^A_\alpha(a_A - \Omega_{AB}a^B)\big] = \big(\Lambda^A{}_B + \Omega^{AC}\Lambda^+_{CB}\big)\big(\epsilon^B_\alpha(a_A - \Omega_{AD}a^D)\big)\;. \tag{4.9}$$

The part of U($2n$) not lying in the Sp(n) subgroup is constrained by the condition

$$\Lambda_{[A}{}^C\Omega_{C|B]} + \Lambda^-_{AB} = 0 \quad\Rightarrow\quad \Lambda_A{}^A = \Omega^{AB}\Lambda^-_{AB}\;. \tag{4.10}$$

The explicit computation (using formulas from appendix B) shows that the associated generators are given by

$$\mathbf{N}^{AB} \equiv \big(a^A + \Omega^{AC}a_C\big)\big(a^B + \Omega^{BD}a_D\big)\;\;,\quad \mathbf{N}_{AB} \equiv \big(a_A - \Omega_{AC}a^C\big)\big(a_B - \Omega_{BD}a^D\big) \tag{4.11}$$

so that $\Omega^{AC}\Omega^{BD}\mathbf{N}_{CD} = \mathbf{N}^{AB}$. Using

$$\big\{\, a_A - \Omega_{AC}a^C\,,\; a_B - \Omega_{BD}a^D\,\big\} = 0\,, \tag{4.12}$$

one easily checks that this particular combination of compact and non-compact generators is *nilpotent*:

$$\big[\mathbf{N}_{AB}, \mathbf{N}_{CD}\big] = \big[\mathbf{N}_{AB}, \mathbf{N}^{CD}\big] = \big[\mathbf{N}^{AB}, \mathbf{N}^{CD}\big] = 0 \tag{4.13}$$

and that the associated Lie algebra elements transform in the $\mathbf{n(2n-1)}$ of $\mathrm{SU}^*(2n)$, viz.

$$\begin{aligned}
\big[\mathbf{X}(\Lambda)\,,\; v^-_{AB}\mathbf{N}^{AB} - v^{-\,AB}\mathbf{N}_{AB}\big] &= 2\big(-\Lambda^C{}_A + \Omega_{AD}\Lambda^{+\,DC}\big)v^-_{CB}\mathbf{N}^{AB}\\
&\quad - 2\big(\Lambda^A{}_C + \Omega^{AD}\Lambda^+_{DC}\big)v^{-\,CB}\mathbf{N}_{AB}\;.
\end{aligned} \tag{4.14}$$

[18] $\mathrm{SU}^*(2n) \cong \mathrm{SL}(n,\mathbb{H})$.

	$\mathcal{N}=2$	$\mathcal{N}=3$	$\mathcal{N}=4$	$\mathcal{N}=5$
$\mathfrak{H}_4$	U(1)	U(3)	U(4)	U(5)
$\mathfrak{J}_1(\mathcal{N})$	$\mathbb{R}$	IcU(2)	Ic(SO(2) × SO(4))	Ic(U(1) × SU(2) × SU(3))
$\mathfrak{J}_2(\mathcal{N})$			ISO(5, 1)	$\big(\mathrm{U}(1)\times\mathrm{Spin}(5,1)\big)\ltimes(S_+\oplus V)$

Table 4. Isotropy subgroups $\mathfrak{J}_n(\mathcal{N}) \subset \mathfrak{H}^*$ for pure $\mathcal{N} \leq 5$ supergravities.

The *reducibility* of the two-form representation of $\mathrm{SU}^*(2n)$ is a direct consequence of the pseudo-reality of its fundamental representation.

The only remaining generators of the $\frac{n}{\mathcal{N}}$ BPS isotropy subgroup (besides the generators $[a^{\bar{A}}, a_{\bar{B}}]$ of $\mathrm{U}(\mathcal{N}-2n)$) correspond to the solutions of

$$\Lambda_{\bar{A}}{}^{C}\Omega_{CB} + \Lambda_{\bar{A}B} = 0 \tag{4.15}$$

in the complex $\mathbf{2n}\otimes(\mathcal{N}-\mathbf{2n})$ representation of $\mathrm{SU}^*(2n)\times\mathrm{U}(\mathcal{N}-2n)$. In terms of fermionic oscillators, the associated generators are

$$\mathbf{N}^{\bar{A}B} = a^{\bar{A}}\big(a^B + \Omega^{BC}a_C\big) \quad , \qquad \mathbf{N}_{\bar{A}B} = a_{\bar{A}}\big(a_B - \Omega_{BC}a^C\big) \ . \tag{4.16}$$

This is once again a combination of compact and non-compact generators, which commutes to give the nilpotent generators in the $\mathbf{n(2n-1)}$ of $\mathrm{SU}^*(2n)$ given above, viz.

$$\begin{aligned} \big[\mathbf{N}_{\bar{A}B}\,,\,\mathbf{N}^{\bar{C}D}\big] &= \delta^{\bar{C}}_{\bar{A}}\,\Omega_{BE}\,\mathbf{N}^{ED} & \big[\mathbf{N}_{\bar{A}B}\,,\,\mathbf{N}_{\bar{C}D}\big] &= 0 \\ \big[\mathbf{N}_{\bar{A}B}\,,\,\mathbf{N}^{CD}\big] &= 0 & \big[\mathbf{N}^{\bar{A}B}\,,\,\mathbf{N}^{CD}\big] &= 0 \ . \end{aligned} \tag{4.17}$$

We thus arrive at the conclusion that *the isotropy subgroups $\mathfrak{J}_n(\mathcal{N})$ are non-reductive subgroups of* $\mathrm{U}(1)\times\mathrm{Spin}^*(2\mathcal{N})$ for $\mathcal{N}\leq 5$, that is, 'Poincaré-like' groups with the product $\mathrm{SU}^*(2n)\times\mathrm{U}(\mathcal{N}-2n)$ as the semi-simple 'Lorentz-like' subgroups; schematically, we have

$$\mathfrak{J}_n(\mathcal{N}) = \big(\mathrm{SU}^*(2n)\times\mathrm{U}(\mathcal{N}-2n)\big)\ltimes\Big(\,(\square\otimes\boxdot)\oplus\begin{smallmatrix}\square\\ \square\end{smallmatrix}_{-}\otimes\mathbf{1}\,\Big)\,, \tag{4.18}$$

where the Young tableaux of $\mathrm{SU}^*(2n)$ and $\mathrm{U}(\mathcal{N}-2n)$ are to be built with undotted and dotted boxes, respectively. The $\frac{n}{\mathcal{N}}$ isotropy subgroup of $\mathrm{U}(1)\times\mathrm{Spin}^*(2\mathcal{N})$ (for $n\geq 1$) is thus of dimension $\mathcal{N}^2+(2n+1)(n-1)$. As we will see below, similar statements hold for $\mathcal{N}=6$ and $\mathcal{N}=8$.

From equation (4.12) it follows that the 'Heisenberg-like' subgroup of the isotropy subgroup leaves invariant the Killing spinor $\epsilon^A_\alpha = -\Omega^{AB}\varepsilon_{\alpha\beta}\epsilon^\beta_B$. Therefore the isotropy subgroup acts on the Killing spinors in the fundamental representation of $\mathrm{SU}^*(2n)$. The isotropy subgroups $\mathfrak{J}_n(\mathcal{N})\subset\mathfrak{H}^*$ for $\mathcal{N}\leq 5$ are given in the table 4; we omit the extra-index on $\mathfrak{J}_n(\mathcal{N})$ as there is only one such group for each pair $(n,\mathcal{N})$ for $\mathcal{N}\leq 5$.
We write IG for the semidirect product of the group G with the abelian translation group in the fundamental representation of G, and IcG is defined to be the semidirect product of the group G with the Heisenberg group defined as the translation group in the fundamental representation of G with a central charge.

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4.2 $\mathcal{N}=6$ supergravity

This description is valid for $\mathcal{N}=6$ supergravity if one restricts to the $\mathrm{U}(1)\times\mathrm{Spin}^*(12)$-orbits of solutions for which the charge matrix satisfies the pure spinor condition. However the decomposition into $\mathrm{SL}(2,\mathbb{R})\times\mathrm{Spin}^*(12)$-orbits of the solutions is more involved and requires one to consider BPS degrees with respect to both $\mathcal{N}=6$ supergravity and the quaternionic $\mathcal{N}=2$ magic supergravity, as well as the vanishing of the horizon area. Indeed the invariance of the extremality parameter $\varkappa\equiv\sqrt{c^2-a^2}$ with respect to $\mathrm{SL}(2,\mathbb{R})\times\mathrm{Spin}^*(12)$ implies that the condition for the horizon area to vanish is left invariant by $\mathrm{SL}(2,\mathbb{R})\times\mathrm{Spin}^*(12)$.

The representation under $\mathrm{SL}(2,\mathbb{R})\times\mathrm{Spin}^*(12)$ of the $\mathcal{N}=6$ charge matrix can be conveniently described by the state (3.22)

$$|\mathscr{C}\rangle=\Big(1+a^7a^8\star\Big)\Big(W+Z_{ij}a^ia^j+\frac{1}{4!}\varepsilon_{ijklmn}\,\Sigma^{mn}a^ia^ja^ka^l+\frac{1}{6!}\varepsilon_{ijklmn}\,Za^ia^ja^ka^la^ma^n\Big)\,|0\rangle \tag{4.19}$$

where $\star$ is the $\mathrm{Spin}^*(12)$ anti-involution defined on chiral spinors. The action of $\mathfrak{spin}^*(12)$ on $\mathscr{C}$ is defined as for lower $\mathcal{N}$, and the action of $\mathfrak{sl}(2,\mathbb{R})$ is defined as follows

$$\delta\,|\mathscr{C}\rangle=\frac{1}{2}\Big(i\lambda\big(a^7a_7+a^8a_8-1\big)+\xi a^7a^8-\bar{\xi}a_7a_8\Big)\,|\mathscr{C}\rangle\ . \tag{4.20}$$

Using the explicit form of the state, one gets

$$\delta\,|\mathscr{C}\rangle=\frac{1}{2}\bigg(\Big(1+a^7a^8\star\Big)(-i\lambda)+\Big(\star+a^7a^8\Big)\xi\bigg)\bigg(W+Z_{ij}a^ia^j+\frac{1}{4!}\varepsilon_{ijklmn}\,\Sigma^{mn}a^ia^ja^ka^l+\frac{1}{6!}\varepsilon_{ijklmn}\,Za^ia^ja^ka^la^ma^n\bigg)\,|0\rangle \tag{4.21}$$

where we used the fact that $\star$ is an *anti*-involution to exhibit the fact that the $\mathrm{U}(1)$ factor λ acts as in the lower $\mathcal{N}$ cases. Let us consider first the non-BPS solutions with a non-vanishing horizon area that would be $\frac{1}{2}$ BPS in $\mathcal{N}=2$ magic supergravity. In this case, the state $|\mathscr{C}\rangle$ can be moved to a basis in which

$$|\mathscr{C}\rangle=\Big(1+a^7a^8\Big)\Big(1+\frac{1}{6!}\varepsilon_{ijklmn}a^ia^ja^ka^la^ma^n\Big)\,|0\rangle\ . \tag{4.22}$$

There is then no way that the generators of $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{spin}^*(12)$ can cancel against each other. The only solution for generators of $\mathfrak{sl}(2,\mathbb{R})$ is given by

$$\xi=-i\lambda \tag{4.23}$$

and by the traceless $\Lambda_i{}^j$ for $\mathfrak{spin}^*(12)$. The isotropy subgroup is thus given in this case by

$$\mathfrak{J}_{(0,1)}(6)\cong\mathbb{R}\times\mathrm{SU}(6)\ . \tag{4.24}$$

For solutions with a non-vanishing horizon area which are $\frac{1}{6}$ BPS with $\mathcal{N}=6$ supergravity, but not BPS in $\mathcal{N}=2$ magic supergravity, the state $|\mathscr{C}\rangle$ can be rotated to a basis in which

$$|\mathscr{C}\rangle=\Big(1+a^7a^8\star\Big)\Big(1+\frac{1}{2}\Omega_{AB}a^Aa^B\Big)\,|0\rangle \tag{4.25}$$

where Ω_{AB} defines a symplectic form on a subspace $\mathbb{C}^2$ of $\mathbb{C}^6$. In this case, the non-compact generators of $\mathfrak{sl}(2,\mathbb{R})$ must be zero in order to leave the state invariant. The computation of the isotropy subgroup is in fact identical to the case of lower $\mathcal{N}$ and one obtains

$$\begin{aligned}\mathfrak{J}_{(1,0)}(6) &\cong \big(\mathrm{SU}(2)\times\mathrm{U}(4)\big)\ltimes\big(\,\square\otimes\boxdot\oplus\mathbb{R}\,\big)\\ &\equiv Ic\big(\mathrm{SU}(2)\times\mathrm{U}(4)\big)\ .\end{aligned}\tag{4.26}$$

For solutions which are $\frac{1}{2}$ BPS in both $\mathcal{N}=2$ and $\mathcal{N}=6$, and for which the horizon area thus necessarily vanishes, the state takes the form

$$\begin{aligned}|\mathscr{C}\rangle &= \Big(1+a^7a^8\Big)\big(1+\star\big)\Big(1+\frac{1}{2}\Omega_{AB}a^Aa^B\Big)\,|0\rangle\\ &= \Big(1+a^7a^8\Big)\Big(1+\frac{1}{2}\Omega_{AB}a^Aa^B\Big)\Big(1+\frac{1}{4!}\varepsilon_{\bar{A}\bar{B}\bar{C}\bar{D}}a^{\bar{A}}a^{\bar{B}}a^{\bar{C}}a^{\bar{D}}\Big)\,|0\rangle\ .\end{aligned}\tag{4.27}$$

Using the same arguments as in the case of lower $\mathcal{N}$, one derives the isotropy subgroup associated to $\mathcal{N}=2$, i.e. $\mathbb{R}$, and the isotropy subgroup associated to $\mathcal{N}=6$, i.e. $Ic\big(\mathrm{SU}(2)\times\mathrm{SU}(4)\big)$. However, the self-duality property of the state implies that the constraint on $\Lambda_{\bar{A}\bar{B}}$ reduces to

$$\Lambda_{\bar{A}\bar{B}}-\frac{1}{2}\varepsilon_{\bar{A}\bar{B}\bar{C}\bar{D}}\Lambda^{\bar{C}\bar{D}}=0\tag{4.28}$$

in such a way that the $\mathrm{SU}(4)\cong\mathrm{Spin}(6)$ factor is enlarged to $\mathrm{Spin}(6,1)$. The product representation of the fundamental of $\mathrm{SU}(2)$ and $\mathrm{SU}(4)$ is promoted to the $\mathrm{SU}(2)$-Majorana representation of $\mathrm{Spin}(6,1)$. The remaining generators of the isotropy subgroup correspond to mixed non-compact transformations of $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{spin}^*(12)$. Indeed, one can compute that

$$\begin{aligned}\big(a^7a^8-a_7a_8\big)\Big(1+a^7a^8\Big)|0\rangle &= \Big(1+a^7a^8\Big)|0\rangle\\ \frac{1}{2}\big(\Omega_{AB}a^Aa^B-\Omega^{AB}a_Aa_B\big)\Big(1+\frac{1}{2}\Omega_{AB}a^Aa^B\Big)|0\rangle &= \Big(1+\frac{1}{2}\Omega_{AB}a^Aa^B\Big)|0\rangle\end{aligned}\tag{4.29}$$

in such a way that

$$\Big(a^7a^8-a_7a_8-\frac{1}{2}\Omega_{AB}a^Aa^B+\frac{1}{2}\Omega^{AB}a_Aa_B\Big)|\mathscr{C}\rangle=0\ .\tag{4.30}$$

The commutation relation of this generator with the nilpotent $\mathbb{R}$ generator gives

$$\begin{aligned}&\Big[a^7a^8-a_7a_8\,,\,ia^7a_7+ia^8a_8-i-ia^7a^8-ia_7a_8\Big]\\ &\qquad\qquad=2\Big(ia^7a_7+ia^8a_8-i-ia^7a^8-ia_7a_8\Big)\end{aligned}\tag{4.31}$$

which defines the Lie algebra of the maximal parabolic subgroup $IGL_+(\mathbb{R})$ of $\mathrm{SL}(2,\mathbb{R})$. This generator commutes with all the other generators, in such a way that the $\frac{1}{6}$ BPS orbit of solutions that are $\frac{1}{2}$ BPS in $\mathcal{N}=2$ magic supergravity is

$$\mathfrak{J}_{(1,1)}(6)\cong IGL_+(\mathbb{R})\ltimes Ic\big(\mathrm{SU}(2)\times\mathrm{Spin}(6,1)\big)\ .\tag{4.32}$$

As discussed in the preceding section, the charge Z is constrained to be a function of the others for solutions of $\mathcal{N} = 6$ supergravity that preserve at least one third of the supersymmetry charges, and $|\mathscr{C}\rangle$ is then a $\mathrm{Spin}^*(12)$ pure spinor. Therefore, we have only one orbit to consider for the $\frac{1}{3}$ and $\frac{1}{2}$ BPS solutions. A $\frac{1}{3}$ BPS charge matrix can be transformed to

$$\mathscr{C} = \Bigl(1 + a^7 a^8 \star\Bigr)\, e^{\frac{1}{2}\Omega_{AB} a^A a^B}\, |0\rangle\,, \tag{4.33}$$

where Ω_{AB} defines a symplectic form over $\mathbb{C}^4$. The same analysis as for lower $\mathcal{N}$ gives that the subgroup

$$\bigl(\mathrm{SU}^*(4) \times \mathrm{SU}(2)\bigr) \ltimes \Bigl(\, \square \otimes \boxdot \oplus \begin{smallmatrix}\square\\ \square\end{smallmatrix}_- \otimes \mathbf{1}\,\Bigr) \subset \mathrm{Spin}^*(12) \tag{4.34}$$

is in the isotropy subgroup. None of the generators of $\mathfrak{sl}(2,\mathbb{R})$ leave the charge matrix invariant on their own. However, let us consider the generators of $\mathfrak{sl}(2,\mathbb{R})$ of the $\mathrm{Spin}^*(4) \cong \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ subgroup of $\mathrm{Spin}^*(12)$ acting on the $\mathbb{C}^2$ subspace orthogonal to the symplectic form Ω_{AB}. Let $\Omega_{\bar{A}\bar{B}}$ be the $\mathrm{SU}(2)$ symplectic form on this subspace; since

$$\frac{1}{2}\Omega_{\bar{A}\bar{B}}\, \frac{1}{8}\Omega_{[AB}\Omega_{CD]} = \frac{1}{6!}\varepsilon_{\bar{A}\bar{B}ABCD}\,, \tag{4.35}$$

we have that

$$\frac{1}{2}\Omega_{\bar{A}\bar{B}} a^{\bar{A}} a^{\bar{B}}\, e^{\frac{1}{2}\Omega_{AB} a^A a^B}\, |0\rangle = \star\, e^{\frac{1}{2}\Omega_{AB} a^A a^B}\, |0\rangle \tag{4.36}$$

and the generators of $\mathfrak{sl}(2,\mathbb{R}) \subset \mathrm{Spin}^*(4)$ act on $|\mathscr{C}\rangle$ as follows

$$\begin{aligned} \frac{1}{2}\Bigl(ib\bigl(a^{\bar{A}} a_{\bar{A}} - 1\bigr) + \frac{1}{2}\zeta\Omega_{\bar{A}\bar{B}} a^{\bar{A}} a^{\bar{B}} - \frac{1}{2}\bar{\zeta}\Omega^{\bar{A}\bar{B}} a_{\bar{A}} a_{\bar{B}}\Bigr)\, e^{\frac{1}{2}\Omega_{AB} a^A a^B}\, |0\rangle& \\ = \frac{1}{2}\Bigl(-ib + \zeta\star\Bigr)\, e^{\frac{1}{2}\Omega_{AB} a^A a^B}\, |0\rangle& \end{aligned} \tag{4.37}$$

i.e. in the same way as do the generators of $\mathfrak{sl}(2,\mathbb{R})$ (4.21). The isotropy subgroup thus also contains the diagonal subgroup of these two $\mathrm{SL}(2,\mathbb{R})$ subgroups. Since one of these $\mathrm{SL}(2,\mathbb{R})$ groups lies in $\mathrm{Spin}^*(4)$, the nilpotent generators that transform in the fundamental of $\mathrm{SU}(2)$ also transform in the fundamental of $\mathrm{SL}(2,\mathbb{R})$. The isotropy subgroup of $\frac{1}{3}$ BPS solutions of $\mathcal{N} = 6$ supergravity is thus

$$\mathfrak{J}_{(2,0)}(6) \cong \bigl(\mathrm{SU}^*(4) \times \mathrm{SO}^*(4)\bigr) \ltimes \Bigl(\, \square \otimes \boxdot \oplus \begin{smallmatrix}\square\\ \square\end{smallmatrix}_- \otimes \mathbf{1}\,\Bigr)\,. \tag{4.38}$$

For the $\frac{1}{2}$ BPS solutions of $\mathcal{N} = 6$ supergravity, the charge matrix can be transformed to

$$\mathscr{C} = \Bigl(1 + a^7 a^8\Bigr)\, e^{\frac{1}{2}\Omega_{AB} a^A a^B}\, |0\rangle \tag{4.39}$$

where Ω_{AB} is a symplectic form on $\mathbb{C}^6$. Again, the same analysis as for lower $\mathcal{N}$ gives the subgroup

$$\mathrm{SU}^*(6) \ltimes \begin{smallmatrix}\square\\ \square\end{smallmatrix}_- \subset \mathrm{Spin}^*(12) \tag{4.40}$$

and the subgroup $\mathbb{R} \subset \mathrm{SL}(2,\mathbb{R})$. Then, following the same argument as for the $\frac{1}{6}$ BPS solutions, one obtains the mixed solution

$$\Bigl(a^7 a^8 - a_7 a_8 - \frac{1}{2}\Omega_{AB} a^A a^B + \frac{1}{2}\Omega^{AB} a_A a_B\Bigr)|\mathscr{C}\rangle = 0\,, \tag{4.41}$$

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dim	0-BPS / $\mathcal{N}=2$ & $\frac{n}{6}$-BPS / $\mathcal{N}=6$	$\frac{1}{2}$-BPS / $\mathcal{N}=2$ & $\frac{n-1}{6}$-BPS / $\mathcal{N}=6$
36	$\mathrm{U}(6)$	
36	$Ic(\mathrm{SU}(2)\times\mathrm{U}(4))$	$\mathbb{R}\times\mathrm{SU}(6)$
37	$IGL_+(1,\mathbb{R})\ltimes Ic(\mathrm{SU}(2)\times\mathrm{Sp}(2)\ltimes\boxminus_+)$	$IGL_+(1,\mathbb{R})\ltimes\mathrm{Sp}(3)\ltimes\boxminus_+$
43	$\big(\mathrm{SU}^*(4)\times\mathrm{SO}^*(4)\big)\ltimes\big(\,\square\otimes\boxdot\oplus\boxminus_-\otimes\mathbf{1}\,\big)$	$IGL_+(\mathbb{R})\ltimes Ic(\mathrm{SU}(2)\times\mathrm{Spin}(6,1))$
52		$IGL_+(\mathbb{R})\ltimes\mathrm{SU}^*(6)\ltimes\boxminus_-$

Table 5. Isotropy subgroups $\mathfrak{J}_{(n-i,i)}(6)\subset\mathrm{SL}(2,\mathbb{R})\times\mathrm{Spin}^*(12)$ in $\mathcal{N}=6$ supergravity.

which defines together with the translation generator the non-semi-simple group $IGL_+(\mathbb{R})$. The isotropy subgroup of the $\frac{1}{2}$ BPS solutions is

$$\mathfrak{J}_{(3,1)}(6)\cong IGL_+(\mathbb{R})\ltimes\mathrm{SU}^*(6)\ltimes\boxminus_-\ . \tag{4.42}$$

So far, we have only discussed the isotropy subgroups associated to the various BPS solutions represented by simple charges for which all the central charges that are not saturated vanish (either $|z_m|=|W|$ or $z_m=0$). However, some solutions define different orbits. This is the case, for instance, for the solutions that are either $\frac{1}{6}$ BPS in the $\mathcal{N}=6$ theory or $\frac{1}{2}$ BPS in the corresponding $\mathcal{N}=2$ theory and which have, moreover, a vanishing horizon area. The horizon area can be computed for such BPS solutions by embedding them into maximal supergravity and then using the conjectured formula for the horizon area of BPS black holes (3.32). In these cases, the computations shows that the corresponding isotropy subgroups $\mathfrak{J}_{(0,1)^\circ}(6)$ and $\mathfrak{J}_{(1,0)^\circ}(6)$ contain an extra $\mathbb{R}_+^*$ factor with respect to the generic ones $\mathfrak{J}_{(0,1)}(6)$ and $\mathfrak{J}_{(1,0)}(6)$, and that some compact generators become nilpotent. We do not consider solutions for which the $E_{7(7)}$ invariant is negative valued, since the energy is negative in this case and all the solutions of the corresponding orbit have naked singularities [35].

The $\mathcal{N}=6$ isotropy subgroups are displayed in table 5.

4.3 $\mathcal{N}=8$ supergravity

The arguments work the same way in the case of maximal supergravity. Let us first discuss the $\frac{1}{2}$ BPS solutions. Using a $\mathrm{U}(8)\subset\mathrm{Spin}^*(16)$ transformation, one can always reach a charge matrix such that W and Z_{ij} are real and such that $Z_{ij}=\frac{W}{2}\Omega_{ij}$, where Ω_{ij} defines a symplectic matrix of $\mathbb{C}^8$. Using a non-compact element of $\mathrm{Spin}^*(16)$ one can then fix W to 1. As for the $\mathcal{N}=1$ to 5 cases, the 0-form, the 2-form and the 4-form components of $|\mathscr{C}\rangle$ then match with $e^{\frac{1}{2}\Omega_{ij}a^ia^j}|0\rangle$. Moreover, because Ω_{ij} defines a real symplectic form of $\mathbb{C}^8$, $e^{\frac{1}{2}\Omega_{ij}a^ia^j}|0\rangle$ is real with respect to the anti-involution $\star$ and matches with $|\mathscr{C}\rangle$ for all form-degree components.

$$|\mathscr{C}\rangle=\exp\left(\frac{1}{2}\Omega_{ij}a^ia^j\right)|0\rangle\ . \tag{4.43}$$

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The computation of the isotropy subgroup works as for lower $\mathcal{N}$, except that there is no extra U(1) generator. The $\frac{1}{2}$ BPS isotropy subgroup is

$$\mathfrak{J}_4(8) = \mathrm{SU}^*(8) \ltimes \boxed{}_{-} \,, \tag{4.44}$$

which is again non-reductive, with the Lorentz-like subgroup $\mathrm{SU}^*(8)$ acting on 28 translations $\mathbb{R}^{28}$; the latter antisymmetric rank-two tensor representation is again real for $\mathrm{SU}^*(8)$ (but not for SU(8)).

As discussed in the preceding section, there is no $\frac{3}{8}$ BPS stationary solution in $\mathcal{N}=8$ supergravity. For both the $\frac{1}{8}$ and $\frac{1}{4}$ solutions, one can reach a basis such that $W=1$ and $2Z_{ij}$ defines a symplectic form on a $\mathbb{C}^2$, respectively $\mathbb{C}^4$, subspace of $\mathbb{C}^8$. In these cases, $\star e^{\frac{1}{2}\Omega_{AB}a^Aa^B}\left|0\right\rangle$ only involves the creation operators $a^{\bar{A}}$ in such a way that it is orthogonal to $e^{\frac{1}{2}\Omega_{AB}a^Aa^B}\left|0\right\rangle$. Thus

$$\left|\mathscr{C}\right\rangle = (1+\star)\, e^{\frac{1}{2}\Omega_{AB}a^Aa^B}\left|0\right\rangle \ . \tag{4.45}$$

By definition, the generators of $\mathfrak{spin}^*(16)$ commute with the involution $\star$, and one gets that the variation of the Majorana–Weyl spinor $\left|\mathscr{C}\right\rangle$ is given by

$$\begin{aligned}\delta\left|\mathscr{C}\right\rangle = (1+\star)\Big(&\big(2\,\Lambda_A{}^C\Omega_{CB} + \Lambda_{AB} + \Omega_{AC}\,\Lambda^{CD}\Omega_{DB}\big)a^Aa^B + \Lambda_{\bar{A}\bar{B}}a^{\bar{A}}a^{\bar{B}} \\ &+2\big(\Lambda_{\bar{A}}{}^C\Omega_{CB} + \Lambda_{\bar{A}B}\big)a^{\bar{A}}a^B + \Omega_{AB}\Lambda^{AB} - \Lambda_A{}^A - \Lambda_{\bar{A}}{}^{\bar{A}}\Big)e^{\frac{1}{2}\Omega_{EF}a^Ea^F}\left|0\right\rangle \ .\end{aligned} \tag{4.46}$$

In the case of the $\frac{1}{4}$ BPS orbit,

$$\Omega_{[AB}\Omega_{CD]} = \frac{1}{3}\varepsilon_{ABCD}\,, \tag{4.47}$$

where ε_{ABCD} defines the $\mathrm{SL}(4,\mathbb{C})$ invariant epsilon tensor. By counting the degree of the various components with respect to the decomposition under $\mathrm{U}(4)\times\mathrm{U}(4)\subset\mathrm{U}(8)$ one obtains that the only components for which the operator $(1+\star)$ introduces a further mixing are

$$(1+\star)\,\Lambda_{\bar{A}\bar{B}}a^{\bar{A}}a^{\bar{B}}\,e^{\frac{1}{2}\Omega_{EF}a^Ea^F}\left|0\right\rangle \ . \tag{4.48}$$

Then using the fact that

$$\star\ \varepsilon_{ABCD}a^Aa^Ba^Ca^D\left|0\right\rangle = \varepsilon_{\bar{A}\bar{B}\bar{C}\bar{D}}a^{\bar{A}}a^{\bar{B}}a^{\bar{C}}a^{\bar{C}}\left|0\right\rangle \,, \tag{4.49}$$

it follows that the condition $\delta\left|\mathscr{C}\right\rangle = 0$ gives the equations

$$\begin{aligned} 2\,\Lambda_{[A}{}^C\Omega_{C|B]} + \Lambda_{AB} + \Omega_{AC}\,\Lambda^{CD}\Omega_{DB} = 0 \qquad & \Lambda_{\bar{A}\bar{B}} - \frac{1}{2}\varepsilon_{\bar{A}\bar{B}\bar{C}\bar{D}}\Lambda^{\bar{C}\bar{D}} = 0 \\ \Lambda_{\bar{A}}{}^C\Omega_{CB} + \Lambda_{\bar{A}B} = 0 \qquad & \Omega_{AB}\Lambda^{AB} - \Lambda_A{}^A - \Lambda_{\bar{A}}{}^{\bar{A}} = 0 \ . \end{aligned} \tag{4.50}$$

The traceless condition and the condition for the $\mathfrak{su}(4)$ generators of the first U(4) factor to leave invariant the symplectic form imply that the maximal compact subgroup of the $\frac{1}{4}$ BPS isotropy subgroup is $\mathrm{Sp}(2)\times\mathrm{SU}(4)\cong\mathrm{Spin}(5)\times\mathrm{Spin}(6)$. The conditions on the non-compact generators

$$\Lambda_{AB} + \Omega_{AC}\,\Lambda^{CD}\Omega_{DB} = 0 \qquad \Omega_{AB}\Lambda^{AB} = 0 \qquad \Lambda_{\bar{A}\bar{B}} - \frac{1}{2}\varepsilon_{\bar{A}\bar{B}\bar{C}\bar{D}}\Lambda^{\bar{C}\bar{D}} = 0 \tag{4.51}$$

restrict the parameters to lie in the vector representation of SO(5) and SO(6) respectively. The maximal semi-simple subgroup of the $\frac{1}{4}$ BPS isotropy subgroup is thus Spin(5, 1) × Spin(6, 1). As for the lower $\mathcal{N}$ case, the nilpotent generators of the isotropy subgroup lie in the $\mathbf{4}\otimes\mathbf{4}$ complex representation of $SU^*(4)\times SU(4)$ and in the $\mathbf{6}$ representation of $SU^*(4)$. They transform in the $\mathbf{32}$ Majorana–Weyl spinor representation of Spin(5, 1) × Spin(6, 1) and the vector representation of SO(5, 1), respectively. Both the $\mathbf{4}$ Weyl representation of Spin(5, 1) and the $\mathbf{8}$ representation of Spin(6, 1) are pseudo-real, but their respective pseudo-anti-involutions permit one to define a real $\mathbf{32}$ spinor representation of Spin(5, 1) × Spin(6, 1). The $\frac{1}{4}$ BPS isotropy subgroup is

$$\mathfrak{J}_2(8) = \big(\mathrm{Spin}(5,1)\times\mathrm{Spin}(6,1)\big)\ltimes\big(\,(\mathbf{4}\otimes\mathbf{8})_{\mathbb{R}}\oplus\mathbf{6}\otimes\mathbf{1}\,\big)\ . \tag{4.52}$$

In the case of the $\frac{1}{8}$ BPS orbit of non-vanishing horizon area, the actions on the two components $e^{\frac{1}{2}\Omega_{AB}a^Aa^B}\,|0\rangle$ and $\star e^{\frac{1}{2}\Omega_{AB}a^Aa^B}\,|0\rangle$ do not mix, and the equations defining the isotropy subgroup of $\mathrm{Spin}^*(16)$ reduce to equations (4.3), with $\lambda = 0$, since there is no U(1) factor in this case. This slight modification of the equation implies that the $\mathrm{U}(\mathcal{N}-2)$ factor of the isotropy subgroup reduces to $\mathrm{SU}(\mathcal{N}-2)$ for $\mathcal{N}=8$. As a result, one gets that

$$\mathfrak{J}_{1^+}(8) = \big(\mathrm{SU}(2)\times\mathrm{SU}(6)\big)\ltimes\big(\,\square\otimes\boxdot\oplus\mathbf{1}\,\big)\ . \tag{4.53}$$

A representative of a $\frac{1}{8}$ BPS solution with vanishing horizon area can be parametrised by three positive real numbers $0<\rho_1<\rho_2<\rho_3<1$ which satisfy $1+\rho_1-\rho_2-\rho_3=0$, as follows:

$$|\mathscr{C}\rangle = (1+\star)\Big(1+a^1a^2\Big)\Big(1+\rho_1\,a^3a^4+\rho_2\,a^5a^6+\rho_3\,a^7a^8\Big)\ |0\rangle\ . \tag{4.54}$$

The generic $\frac{1}{8}$ BPS isotropy subgroup $\mathfrak{J}_{1^+}(8)$ is not modified by the deformation associated to the parameters ρ_1, ρ_2 and ρ_3 as long as they satisfy $0<\rho_1<\rho_2<\rho_3<1$ and $1+\rho_1-\rho_2-\rho_3>0$. The subgroup $\mathrm{SU}(2)\times\mathrm{Sp}(3)\subset\mathrm{SU}(2)\times\mathrm{SU}(6)$ remains unchanged for any value of $1+\rho_1-\rho_2-\rho_3$, but the signature of the remaining generators with respect with the Cartan form depends on the sign of $1+\rho_1-\rho_2-\rho_3$, in such a way that when the latter is negative, the isotropy subgroup is

$$\mathfrak{J}_{1^-}(8) = \big(\mathrm{SU}(2)\times\mathrm{SU}^*(6)\big)\ltimes\big(\,\square\otimes\boxdot\oplus\mathbf{1}\,\big)\ . \tag{4.55}$$

This corresponds to $\frac{1}{8}$ BPS solutions for which $\lozenge(W^{-\frac{1}{2}}Z)<0$. Such solutions carry a naked singularity and will be disregarded [35]. For $1+\rho_1-\rho_2-\rho_3=0$ most of the generators are nilpotent and there is an extra $\mathbb{R}_+^*$ invariance of the charge matrix which decreases the dimension of the corresponding orbit by one. The isotropy subgroup of the $\frac{1}{8}$ BPS solutions of vanishing horizon area is

$$\begin{aligned}\mathfrak{J}_{1^\circ}(8) &= \big(\mathbb{R}_+^*\times\mathrm{SU}(2)\times\mathrm{Sp}(3)\big)\ltimes\Big(\ \big((\boxdot\otimes\square)_+\oplus\begin{smallmatrix}\square\\\square\end{smallmatrix}_+\big)^{(1)}\oplus(\boxdot\otimes\square)_+^{(2)}\oplus\mathbf{1}^{(3)}\ \Big)\\ &= Ic\big(\mathrm{SU}(2)\times(\mathbb{R}_+^*\times\mathrm{Sp}(3))\ltimes\begin{smallmatrix}\square\\\square\end{smallmatrix}_+\big)\ .\end{aligned} \tag{4.56}$$

5 Orbits of stationary single-particle solutions

Under the action of an element $g \in \mathfrak{G}$, the coset representative $\mathcal{V}$ transforms as

$$\mathcal{V} \quad \rightarrow \quad \mathcal{V}(g) = g\mathcal{V}h(g,\mathcal{V})\,, \tag{5.1}$$

where $h(g,\mathcal{V})$ is the element of $\mathfrak{H}^*$ that permits one to reach the specific representative of the class $[g\mathcal{V}]$ in the chosen parametrization of the coset space $\mathfrak{G}/\mathfrak{H}^*$. The subgroup of $\mathfrak{G}$ preserving the asymptotic flatness condition $\mathcal{V} \rightarrow \mathbb{1}$ is thus $\mathfrak{H}^*$. As we explained in the first section, all the non-extremal asymptotically flat axisymmetric stationary single-particle solutions which are regular outside the horizon are in the $\mathfrak{H}^*$-orbit of some Kerr solution.[19] In the following, we will discuss these orbits in detail for all pure supergravity theories.

In general, both horizon area and surface gravity (hence also the associated thermodynamic quantities, i.e. entropy and temperature) are invariant with respect to the four-dimensional duality group $\mathfrak{G}_4$. However, neither of them is invariant under the action of the three-dimensional group $\mathfrak{H}^*$ since the relevant expressions depend explicitly on the mass and the NUT charge. Nevertheless it has been observed that *the product of the horizon area and the surface gravity* is equal to the deviation from extremality $4\pi\varkappa$ [36], which *is* invariant under the action of $\mathfrak{H}^*$. This statement is still valid for non-extremal multi-black-hole solutions. It turns out that both the horizon area and the surface gravity are modified by the presence of other black holes, but their product remains equal to $4\pi\varkappa$. We should mention that the statistical interpretation of the horizon area and the surface gravity in the case of an asymptotically Taub–NUT solution is not clear [37]. One important fact that follows from this invariance is that the horizon area transforms by a non-linear rescaling with respect to the action $\mathfrak{H}^*$. Therefore, although the horizon area A is generally not invariant with respect to the action of $\mathfrak{H}^*$, the condition $A = 0$ is.

5.1 Stratified structure of the moduli spaces of charges

The $\mathfrak{H}^*$-orbits of single-particle solutions can be characterised in terms of the $\mathfrak{H}^*$-orbits of the charge matrix $\mathscr{C}$ in $\mathfrak{g} \ominus \mathfrak{h}^*$. The decomposition of the set of asymptotically flat axisymmetric stationary single-particle solutions, including the extremal solutions, which can be obtained as special limits of non-extremal ones, can be derived from the decomposition into $\mathfrak{H}^*$-orbits of charge matrices $\mathscr{C}$ satisfying the cubic equation $\mathscr{C}^3 = c^2\mathscr{C}$ or its quintic analogue (2.21). The set of such charge matrices (*alias* the *moduli space of solutions* of (2.22) or (2.21)) is a stratified space $\mathcal{M}$, that is, a partially ordered union of manifolds

$$\mathcal{M} = \bigcup_{n\in I} \mathcal{M}_n\,, \tag{5.2}$$

where the submanifolds $\mathcal{M}_n$, are such that all their intersections are empty, that is, $\mathcal{M}_n \cap \mathcal{M}_m = \emptyset$, and the intersection of the closure of a given stratum $\overline{\mathcal{M}_n}$ with another stratum

[19] We recall that these solutions do not exhaust the full set of stationary solutions to the equations of motion. However, all non-extremal solutions lying off $\mathfrak{H}^*$ orbits passing through regular Kerr solutions are comprised entirely of singular solutions without horizons.

$\mathcal{M}_m$ is either empty or $\mathcal{M}_m$ itself

$$\overline{\mathcal{M}_n} \cap \mathcal{M}_m \neq \emptyset \quad \Rightarrow \quad \mathcal{M}_m \subset \overline{\mathcal{M}_n}\,. \tag{5.3}$$

There is a main stratum $\mathcal{M}_0$, whose closure is $\mathcal{M}$ itself. The stratification is said to be ordered if for any m and n in I, either $\mathcal{M}_m \subset \overline{\mathcal{M}_n}$ or $\mathcal{M}_n \subset \overline{\mathcal{M}_m}$. For an ordered stratification, we label the strata by integers, such that $m > n$ means that $\mathcal{M}_m \subset \overline{\mathcal{M}_n}$.

The main stratum $\mathcal{M}_0$ corresponds to solutions with $c^2 \neq 0$, hence to non-BPS solutions; it has the structure

$$\mathcal{M}_0 \;=\; \mathbb{R}_+^* \times \mathfrak{H}^*/\mathfrak{H}_4\,, \tag{5.4}$$

where the coset $\mathfrak{H}^*/\mathfrak{H}_4$ encodes the gravitational and electromagnetic charges for fixed c^2, and the extra factor $\mathbb{R}_+^*$ corresponds to the non-zero values of the BPS parameter c^2. Clearly, re-scalings of c are *not* part of the group $\mathfrak{H}^*$; however, as we will show in the following section, they are associated to the so-called 'trombone symmetry' [15]. Modulo certain conformal diffeomorphisms, the latter can be incorporated into the full three-dimensional duality group $\mathfrak{G}$, as we will show below.

The other strata $\mathcal{M}_n$ with $n \neq 0$ parametrise solutions with $c^2 = 0$. The charge matrix $\mathscr{C}$ of such strata parametrises stationary non-rotating extremal solutions, like spherically symmetric extremal black holes or multi-black-hole solutions. These strata are $\mathfrak{H}^*$-orbits with

$$\mathcal{M}_n \;\cong\; \mathfrak{H}^*/\mathfrak{J}_n\,, \tag{5.5}$$

where the $\mathfrak{J}_n = \mathfrak{J}_n(\mathscr{C})$ are the isotropy groups that leave invariant the given charge matrix $\mathscr{C}$, and which were analysed in the previous section for pure supergravity. We note that the space of single-particle-like stationary solutions is likewise a stratified space. It differs from the above moduli space of charges only by the extra information not captured by $\mathscr{C}$, namely the value of the angular momentum parameter a, which is restricted to lie in the interval $-c \leq a \leq c$ because we are excluding hyper-extremal solutions (the values $a = \pm c$ give extremal Kerr solutions).

We next show that *each $\mathfrak{H}^*$-orbit in $\mathcal{M}$ is a Lagrangian submanifold* of a $\mathfrak{G}$-orbit space. For this purpose, we define a larger isotropy group $\mathfrak{J}'_n \equiv \mathfrak{J}'_n(\mathscr{C}) \subset \mathfrak{G}$ consisting of all transformations $g \in \mathfrak{G}$ leaving invariant the given charge matrix $\mathscr{C}$; clearly $\mathfrak{J}_n \subset \mathfrak{J}'_n$. To see that the inclusion

$$\mathfrak{H}^*/\mathfrak{J}_n \subset \mathfrak{G}/\mathfrak{J}'_n \tag{5.6}$$

embeds $\mathfrak{H}^*/\mathfrak{J}_n$ as a Lagrangian submanifold we introduce the symplectic form

$$\omega(x,y)\big|_{\mathscr{C}} \equiv \mathrm{Tr}\; \mathscr{C}[\boldsymbol{x},\boldsymbol{y}] \tag{5.7}$$

on $\mathfrak{G}/\mathfrak{J}'_n$. Here, x and y are invariant vector fields $\in T\big(\mathfrak{G}/\mathfrak{J}'_n\big)$ which coincide with the class of Lie algebra elements $[\boldsymbol{x}]$, $[\boldsymbol{y}] \in \mathfrak{g}\,/\,\mathfrak{j}'_n \cong T_{\mathscr{C}}\big(\mathfrak{G}/\mathfrak{J}'_n\big)$ at $\mathscr{C} \in \mathfrak{G}/\mathfrak{J}'_n$;[20] observe that the r.h.s. of (5.7) vanishes when $\boldsymbol{x}$ or $\boldsymbol{y}$ or both are in $\mathfrak{j}'_n$ and thus it is well-defined on $\mathfrak{g}\,/\,\mathfrak{j}'_n$. On a point $\mathscr{C} \in \mathfrak{H}^*/\mathfrak{J}_n \subset \mathfrak{G}/\mathfrak{J}'_n$, since $\mathscr{C} \in \mathfrak{g} \ominus \mathfrak{h}^*$ it follows that, if $[\boldsymbol{x}]$ admits a representative $\boldsymbol{x} \in \mathfrak{h}^*$,

[20] Here $\mathfrak{g}\,/\,\mathfrak{j}'_n$ is the class of elements of $\mathfrak{g}$ that become identified when their difference lies in $\mathfrak{j}'_n$.

the symplectic form $\omega(x,y)\big|_{\mathscr{C}}$ is non-zero only if $[\boldsymbol{y}]$ admits a non-trivial representative $\boldsymbol{y} \in \mathfrak{g} \ominus \mathfrak{h}^*$, which proves that $T_{\mathscr{C}}\big(\mathfrak{H}^*/\mathfrak{J}_n\big) \subset T_{\mathscr{C}}\big(\mathfrak{G}/\mathfrak{J}'_n\big)$ is isotropic with respect with $\omega|_{\mathscr{C}}$. Moreover, for any non-trivial representative $\boldsymbol{y} \in \mathfrak{g} \ominus \mathfrak{h}^*$, $[\mathscr{C}, \boldsymbol{y}]$ is a non-zero element of $\mathfrak{h}^*$ such that there exits $\boldsymbol{x} \in \mathfrak{h}^*$ for which Tr $\mathscr{C}[\boldsymbol{x}, \boldsymbol{y}] \neq 0$ (the existence being ensured by the non-degeneracy of the symplectic form ω). Therefore $T_{\mathscr{C}}\big(\mathfrak{H}^*/\mathfrak{J}_n\big) \subset T_{\mathscr{C}}\big(\mathfrak{G}/\mathfrak{J}'_n\big)$ is Lagrangian with respect with $\omega|_{\mathscr{C}}$. We conclude that $\mathfrak{H}^*/\mathfrak{J}_n$ is a Lagrangian submanifold of $\mathfrak{G}/\mathfrak{J}'_n$ with respect to the symplectic form ω.

It is important to emphasise the link between the moduli spaces $\mathcal{M}_n$ (for $n \geq 1$) and the nilpotent adjoint orbits of the corresponding group, which have been extensively studied by mathematicians [38].[21] This link was already emphasised in [39], and we can now state it in a precise way. Although we are interested in real simple Lie algebras $\mathfrak{g}$, the characterisation of their nilpotent orbits requires one to consider the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. Define $\mathfrak{N}_{\mathfrak{G}_{\mathbb{C}}}$ as the variety of nilpotent elements of $\mathfrak{g}_{\mathbb{C}}$. $\mathfrak{N}_{\mathfrak{G}_{\mathbb{C}}}$ is a stratified space and each stratum is a $\mathfrak{G}_{\mathbb{C}}$-orbit, where $\mathfrak{G}_{\mathbb{C}}$ is the complexification of the simple Lie group $\mathfrak{G}$,

$$\mathfrak{N}_{\mathfrak{G}_{\mathbb{C}}} \cong \bigcup_{n \in I_{\mathfrak{G}}} \frac{\mathfrak{G}_{\mathbb{C}}}{\mathfrak{J}^{(n)}_{\mathfrak{G}_{\mathbb{C}}}} \ . \tag{5.8}$$

where the index-set $I_{\mathfrak{G}}$ labels the different isotropy subgroups and thus the inequivalent orbits. The subspaces

$$\mathfrak{N}_{\mathfrak{G}} \equiv \mathfrak{N}_{\mathfrak{G}_{\mathbb{C}}} \cap \mathfrak{g} \qquad \mathfrak{N}_{\mathfrak{H}_{\mathbb{C}}} \equiv \mathfrak{N}_{\mathfrak{G}_{\mathbb{C}}} \cap (\mathfrak{g}_{\mathbb{C}} \ominus \mathfrak{h}_{\mathbb{C}}) \tag{5.9}$$

are also stratified spaces which decompose into (real) $\mathfrak{G}$-orbits and $\mathfrak{H}_{\mathbb{C}}$-orbits respectively. The Kostant–Sekiguchi correspondence [40] states that their stratifications are identical since there exists a homeomorphism [41]

$$\frac{\mathfrak{N}_{\mathfrak{G}}}{\mathfrak{G}} \cong \frac{\mathfrak{N}_{\mathfrak{H}_{\mathbb{C}}}}{\mathfrak{H}_{\mathbb{C}}} \ . \tag{5.10}$$

Thanks to this homeomorphism, the problem of determining the stratification of the real algebraic variety $\mathfrak{N}_{\mathfrak{G}}$ reduces to the much easier problem of determining the stratification of the complex algebraic variety $\mathfrak{N}_{\mathfrak{H}_{\mathbb{C}}}$.

In supergravity, the charge matrix lies in $\mathfrak{g} \ominus \mathfrak{h}^*$, and we are thus interested in the subvariety $\mathfrak{N}_{\mathfrak{H}^*} \subset \mathfrak{N}_{\mathfrak{G}}$

$$\mathfrak{N}_{\mathfrak{H}^*} \equiv \mathfrak{N}_{\mathfrak{G}} \cap (\mathfrak{g} \ominus \mathfrak{h}^*) \tag{5.11}$$

which defines the moduli space of charge matrices of (possibly singular) extremal spherically symmetric black hole solutions. As we have just proved, $\mathfrak{N}_{\mathfrak{H}^*}$ is in fact a Lagrangian subvariety of $\mathfrak{N}_{\mathfrak{G}}$ in the sense that each $\mathfrak{H}^*$-orbit inside $\mathfrak{N}_{\mathfrak{H}^*}$ is a Lagrangian submanifold of a $\mathfrak{G}$-orbit inside $\mathfrak{N}_{\mathfrak{G}}$. Nevertheless, some $\mathfrak{G}$-orbits of $\mathfrak{N}_{\mathfrak{G}}$ do not contain any $\mathfrak{H}^*$-orbit inside $\mathfrak{N}_{\mathfrak{H}^*}$. The $\mathfrak{H}^*$-orbits inside $\mathfrak{N}_{\mathfrak{H}^*}$ can be classified by a determination of the inequivalent embeddings of $\mathfrak{h}^* \subset \mathfrak{g}$ such that a given representative of the corresponding nilpotent orbit in $\mathfrak{N}_{\mathfrak{G}}$ lies inside $\mathfrak{g} \ominus \mathfrak{h}^*$. In this way, one can compute the isotropy subgroups of $\mathfrak{H}^*$-orbits without knowing explicitly the charge matrix $\mathscr{C}$ of any of its representatives as

[21]We are grateful to B. Pioline for having drawn our attention to [38].

a function of the conserved charges w and Z. As we shall see, this permits one to show the existence of an $\mathfrak{H}^*$-orbit of non-BPS extremal solutions inside $\mathcal{M}$ in both $\mathcal{N}=8$ and $\mathcal{N}=6$ supergravities.

Among the $\mathfrak{G}$ nilpotent orbits, there is a minimal non-trivial nilpotent orbit which is at the boundary of any orbit inside $\mathfrak{N}_{\mathfrak{G}}$. In pure supergravity theories, the minimal $\mathfrak{G}$-orbit (i.e. $\mathfrak{G}/\big(\mathfrak{G}_4 \ltimes (\mathfrak{l}_4 \oplus \mathbb{R})\big)$ in these cases), generically does not contain any $\mathfrak{H}^*$-orbit in $\mathfrak{g} \ominus \mathfrak{h}^*$. Only in $\mathcal{N}=6$ and $\mathcal{N}=8$ supergravities do the respective minimal orbits contain $\mathfrak{H}^*$-orbits of $\frac{1}{2}$ BPS charge matrices. The minimal nilpotent orbits seem to be associated to maximally supersymmetric black holes in general.

Since there is no uniqueness theorem for extremal solutions which would generalise Mazur's theorem for non-extremal solutions, it is natural to enquire whether higher-order orbits of $\mathfrak{N}_{\mathfrak{H}^*}$, which do not lie on the boundary of $\mathcal{M}_0$, can correspond to regular extremal solutions of supergravity. There is no such orbit when the theory contains no scalar fields, but there can be many otherwise.

Pure supergravity. As we have shown in section 3.3, for all supergravity theories with $\mathcal{N} \le 5$, all solutions with a vanishing BPS parameter $c=0$ are BPS and the stratification is ordered with respect to the BPS degree. Indeed, $\mathcal{M}$ is then the space of $\mathrm{Spin}^*(2\mathcal{N})$ pure spinors, which admits the following stratification by BPS degree

$$\mathcal{M}_0 \cong \mathbb{C}^\times \times \frac{\mathrm{Spin}^*(2\mathcal{N})}{\mathrm{U}(\mathcal{N})}\,, \quad \mathcal{M}_n \cong \frac{\mathrm{U}(1)\times\mathrm{Spin}^*(2\mathcal{N})}{\big(\mathrm{SU}^*(2n)\times\mathrm{U}(\mathcal{N}-2n)\big)\ltimes\big(\,(\square\otimes\boxdot)\oplus \substack{\square\\\square}_-\otimes\mathbf{1}\,\big)} \tag{5.12}$$

such that the last stratum is just a single point $\{0\}$ (the trivial solution). The orbits of $\frac{n}{\mathcal{N}}$ BPS stationary solutions are of dimension $\mathcal{N}^2-\mathcal{N}+1-(2n+1)(n-1)$.

The stratification is more involved in the case of $\mathcal{N}=6$ supergravity. In this case $\mathcal{M}_{(p,q)}$ corresponds to solutions which are $\frac{p}{6}$ BPS in $\mathcal{N}=6$ supergravity and $\frac{q}{2}$ BPS in the corresponding magic supergravity associated to the quaternions. $\mathcal{M}_{(p,q)} \subset \overline{\mathcal{M}}_{(r,s)}$ if and only if both $p>r$ and $q>s$, and $\partial\mathcal{M}_{(p,q)} = \overline{\mathcal{M}}_{(p,q)^\circ}$.

$$\begin{aligned}
\mathcal{M}_{(0,0)} &\cong \mathbb{R}_+^*\times\frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{Spin}^*(12)}{\mathrm{U}(6)} \\
\mathcal{M}_{(1,0)} &\cong \frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{Spin}^*(12)}{Ic(\mathrm{SU}(2)\times\mathrm{U}(4))} \qquad\qquad \mathcal{M}_{(0,1)} \cong \frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{Spin}^*(12)}{\mathbb{R}\times\mathrm{SU}(6)} \qquad (5.13)\\
\mathcal{M}_{(1,0)^\circ} &\cong \frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{Spin}^*(12)}{IGL_+(1,\mathbb{R})\ltimes Ic(\mathrm{SU}(2)\times\mathrm{Sp}(2)\ltimes\substack{\square\\\square}_+)} \quad \mathcal{M}_{(0,1)^\circ} \cong \frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{Spin}^*(12)}{IGL_+(1,\mathbb{R})\ltimes\mathrm{Sp}(3)\ltimes\substack{\square\\\square}_+} \\
\mathcal{M}_{(1,1)} &\cong \frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{Spin}^*(12)}{IGL_+(\mathbb{R})\times Ic(\mathrm{SU}(2)\times\mathrm{Spin}(6,1))} \\
\mathcal{M}_{(2,0)} &\cong \frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{Spin}^*(12)}{\big(\mathrm{SU}^*(4)\times\mathrm{SO}^*(4)\big)\ltimes\big(\,\square\otimes\boxdot\oplus\substack{\square\\\square}_-\otimes\mathbf{1}\,\big)} \\
\mathcal{M}_{(3,1)} &\cong \frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{Spin}^*(12)}{IGL_+(\mathbb{R})\times\mathrm{SU}^*(6)\ltimes\substack{\square\\\square}_-}\,.
\end{aligned} \tag{5.14}$$

This stratification is in agreement with the stratification of $\mathfrak{N}_{E_{7(-5)}}$ [42], although the latter suggests that there is an additional stratum $\mathcal{M}_{(0,0)^\circ}$ of dimension 33 in the boundary of

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the main stratum $\mathcal{M}_{(0,0)}$,

$$\mathcal{M}_{(0,0)^\circ} \cong \frac{\mathrm{SL}(2,\mathbb{R}) \times \mathrm{Spin}^*(12)}{\mathrm{Sp}(3) \ltimes \boxminus_+ \times \mathbb{R}} \tag{5.15}$$

whose boundary is

$$\partial\mathcal{M}_{(0,0)^\circ} = \mathcal{M}_{(1,0)^\circ} \cup \mathcal{M}_{(0,1)^\circ} \cup \mathcal{M}_{(1,1)} \cup \mathcal{M}_{(2,0)} \cup \mathcal{M}_{(3,1)} \ . \tag{5.16}$$

This stratum does indeed exist, and corresponds to non-BPS extremal solutions, such as for example the ones discovered in [34] within the STU model. We note also that the first strata (corresponding to elements satisfying $\mathrm{ad}_{\mathscr{e}}{}^5 = 0$) of the nilpotent orbits of $F_{4(4)}$, $E_{6(2)}$ and $E_{8(-24)}$ all have the same stratification ordering as those of $E_{7(-5)}$ [43]. This suggests that the moduli spaces of all four magic $\mathcal{N} = 2$ supergravity theories might have the same stratification, i.e. that the quotients $\mathcal{M}/\mathfrak{H}^*$ associated to these theories might all be homeomorphic.

The moduli space of solutions to the quintic $\mathcal{N} = 8$ characteristic equation decomposes into the strata

$$\begin{aligned}
\mathcal{M}_0 &\cong \mathbb{R}_+^* \times \frac{\mathrm{Spin}^*(16)}{\mathrm{SU}(8)} , \qquad \mathcal{M}_1 \cong \frac{\mathrm{Spin}^*(16)}{Ic(\mathrm{SU}(2) \times \mathrm{SU}(6))} \\
\mathcal{M}_{1^\circ} &\cong \frac{\mathrm{Spin}^*(16)}{Ic\big(\mathrm{SU}(2) \times (\mathbb{R}_+^* \times \mathrm{Sp}(3)) \ltimes \boxminus_+\big)} \\
\mathcal{M}_2 &\cong \frac{\mathrm{Spin}^*(16)}{\big(\mathrm{Spin}(5,1) \times \mathrm{Spin}(6,1)\big) \ltimes \big(\ \mathbf{4} \otimes \mathbf{8} \oplus \mathbf{6} \otimes \mathbf{1}\ \big)} , \qquad \mathcal{M}_4 \cong \frac{\mathrm{Spin}^*(16)}{\mathrm{SU}^*(8) \ltimes \boxminus_-}
\end{aligned} \tag{5.17}$$

together with the trivial solution $\{0\}$. The ordering 0, 1, 1°, 2, 4 is in agreement with the stratification of $\mathfrak{N}_{E_{8(8)}}$ [44], although the latter suggests that there is an additional stratum $\mathcal{M}_{0^\circ}$ of dimension 57 in the boundary of $\mathcal{M}_0$,

$$\mathcal{M}_{0^\circ} \cong \frac{\mathrm{Spin}^*(16)}{\mathrm{Sp}(4) \ltimes \boxminus_-} \tag{5.18}$$

which has the same boundary as $\mathcal{M}_1$. This stratum does indeed exist, and corresponds to non-BPS extremal solutions. None of the central charges of the solutions lying in this orbit is saturated (i.e. $|z_m|^2 < |W|^2$), and they all satisfy $\diamondsuit(W^{-\frac{1}{2}}Z) < 0$.

Let us compare these moduli spaces with the moduli spaces of $\frac{1}{2}$ and $\frac{1}{4}$ BPS static black holes [35] (i.e. with vanishing NUT charge)

$$\begin{aligned}
\mathcal{M}_4^{\text{static}} &\cong \frac{E_{7(7)}}{E_{6(6)} \ltimes \mathbf{27}} \cong \mathbb{R}_+^* \times \frac{\mathrm{SU}(8)}{\mathrm{Sp}(4)} \\
\mathcal{M}_2^{\text{static}} &\cong \frac{E_{7(7)}}{Pin(5,6) \ltimes (\mathbf{32} \oplus \mathbb{R})} \cong \mathbb{R}_+^* \times \mathbb{R}_+ \times \frac{\mathrm{SU}(8)}{\mathrm{Sp}(2) \times \mathrm{Sp}(2)}
\end{aligned} \tag{5.19}$$

where the $E_{7(7)}$ coset spaces correspond to orbits of the active duality group [45]. Note that the active $E_{7(7)}$ transformations on solutions with non-vanishing NUT charge do not preserve the BPS degree in general, so that there is no well-defined action of the active

	$\mathcal{N}=2$	$\mathcal{N}=3$	$\mathcal{N}=4$	$\mathcal{N}=5$	$\mathcal{N}=6$	$\mathcal{N}=8$
$\dim(\mathcal{M}_0)$	4	8	14	22	34	58
$\dim(\mathcal{M}_1)$	3	7	13	21	33	57
$\dim(\mathcal{M}_{1^\circ})$					32	56
$\dim(\mathcal{M}_2)$			8	16	26	46
$\dim(\mathcal{M}_4)$					17	29

Table 6. Dimensions of strata in pure supergravity.

duality group $E_{7(7)}$ on the strata $\mathcal{M}_2$ and $\mathcal{M}_4$. The fact that the action does preserve the BPS degree for static solutions is related to the fact that the $\frac{1}{4}$ BPS condition is associated to the vanishing of the quartic $E_{7(7)}$ invariant $\lozenge(Z)$ for asymptotically Minkowskian solutions, whereas it is related to the vanishing of $\lozenge(W^{-\frac{1}{2}}Z)$ in general. These strata are therefore non-trivial fibre bundles with respect to the Ehlers U(1):

$$\begin{array}{ccc} \mathrm{U}(1) \to \dfrac{\mathrm{Spin}^*(16)}{\mathrm{SU}^*(8)\ltimes\boxminus_-} & \qquad & \mathrm{U}(1) \to \dfrac{\mathrm{Spin}^*(16)}{(\mathrm{Spin}(5,1)\times\mathrm{Spin}(6,1))\ltimes(\,\mathbf{4}\otimes\mathbf{8}\oplus\mathbf{6}\otimes\mathbf{1}\,)} \\ \downarrow & & \downarrow \\ \dfrac{E_{7(7)}}{E_{6(6)}\ltimes\mathbf{27}} & & \dfrac{E_{7(7)}}{Pin(5,6)\ltimes(\mathbf{32}\oplus\mathbb{R})} \end{array} \; . \tag{5.20}$$

It follows that there is no action of $E_{7(7)}$ on $\mathcal{M}_2$ and $\mathcal{M}_4$ that would agree on a fixed SU(8) subgroup, with the action of $\mathrm{Spin}^*(16)$. In fact, this would be inconsistent since their closure would then generate a well-defined action of $E_{8(8)}$ on the 29 (respectively 46) dimensional strata whereas the minimal representation of $E_{8(8)}$ is 57-dimensional [46]. Although there is no 29-dimensional representation of $E_{8(8)}$, the minimal unitary representation of $E_{8(8)}$ acts on the space of functions defined on a 29-dimensional Lagrangian submanifold of the 56-dimensional minimal adjoint orbit [20, 21], which we have just proved to be diffeomorphic to $\mathcal{M}_4$. We will come back to this observation when we discuss the nilpotency degree of the charge matrix on each stratum.

The dimensions of the various strata of pure supergravity theories are summarised in table 6.

It follows from the cubic equation (or its quintic analogue) that a charge matrix of $\mathcal{M}_1$ satisfies $\mathscr{C}^3 = 0$ (or $\mathscr{C}^5 = 0$ for E_8). It turns out that the order of the stratum n is related to the nilpotency degree of the charge matrix in general and thus that for pure supergravity theories, the BPS degree of the solutions is characterised in a $\mathfrak{G}$ invariant way by the nilpotency degree of the charge matrix. For $\mathcal{N}=2,3$ the condition $\mathscr{C}^2 = 0$ implies that the charge matrix vanishes and that $\mathcal{M}_1$ is the last non-trivial stratum. As we will see in section 6, for $\mathcal{N}=4$ supergravity, $\mathscr{C}^2 = 0$ on $\mathcal{M}_2$. To summarise briefly, we have for low values of $\mathcal{N}$ that
For $\mathcal{N} \geq 5$ supergravity, the nilpotency degree in the fundamental representation of $\mathfrak{e}_{6(-14)}$, $\mathfrak{e}_{7(-5)}$ or $\mathfrak{e}_{8(8)}$ is not enough to characterise the degree of the strata. It is then

	$\mathcal{N}=2$	$\mathcal{N}=3$	$\mathcal{N}=4$
$\mathcal{M}_1$	$\mathscr{C}^3=0$	$\mathscr{C}^3=0$	$\mathscr{C}^3=0$
$\mathcal{M}_2$			$\mathscr{C}^2=0$

Table 7. Nilpotency degree of charge matrices for $\mathcal{N}=2$, 3, 4.

useful to consider $\mathcal{N}=4$ supergravity as a consistent truncation of $\mathcal{N}=5$ supergravity, both of them as consistent truncations of $\mathcal{N}=6$ supergravity, and all three of them as consistent truncations of $\mathcal{N}=8$ supergravity. These truncations can be understood from the decompositions of $\mathfrak{e}_{8(8)}$

$$\begin{aligned}\mathfrak{e}_{8(8)} &\cong \mathfrak{su}(2)\oplus\mathfrak{e}_{7(-5)}\oplus\big(\mathbf{2}\otimes\mathbf{56}\big)_{\mathbb{R}} \\ &\cong \mathfrak{su}(2)\oplus\big(\mathfrak{u}(1)\oplus\mathfrak{e}_{6(-14)}\oplus\mathbf{27}\big)\oplus\big(\mathbf{2}\oplus\mathbf{2}\otimes\mathbf{27}\big) \\ &\cong \mathfrak{su}(2)\oplus\mathfrak{u}(1)\oplus\big(\mathfrak{u}(1)\oplus\mathfrak{so}(2,8)\oplus\mathbf{16}_+\big)\oplus\big(\mathbf{10}\oplus\mathbf{16}_-\oplus\mathbf{1}\big)\oplus\mathbf{2}\oplus\mathbf{2}\otimes\big(\mathbf{10}\oplus\mathbf{16}_-\oplus\mathbf{1}\big)\end{aligned} \tag{5.21}$$

where the representations are complex when unspecified. It follows that a solution of $\mathcal{N}=5,6$ supergravity, corresponding upon embedding into $\mathcal{N}=8$ supergravity to a solution with an $\mathfrak{e}_{8(8)}$ charge matrix satisfying $\mathscr{C}^n=0$, has an $\mathfrak{e}_{6(-14)}$ or $\mathfrak{e}_{7(-5)}$ charge matrix that satisfies both

$$\mathscr{C}^n=0 \quad \text{and} \quad \mathrm{ad}_{\mathscr{C}}{}^n=0\ . \tag{5.22}$$

The condition $\mathscr{C}^3=0$ on $\mathcal{M}_1$ implies $\mathrm{ad}_{\mathscr{C}}{}^5=0$. For $\frac{1}{4}$ BPS solutions in $\mathcal{N}=8$ supergravity, it is convenient to consider the case for which they can be understood as $\frac{1}{2}$ BPS solutions in $\mathcal{N}=4$ supergravity. The $\mathfrak{spin}(2,8)$ charge matrix then satisfies $\mathscr{C}^2=0$ in the spinor representations, which implies $\mathrm{ad}_{\mathscr{C}}{}^3=0$. However, one checks that the charge matrix is not nilpotent in the vector representation $[\mathscr{C},[\mathscr{C},\Gamma^{\mathcal{M}}]]\neq 0$. Since the fundamental representation of $E_{6(-14)}$ decomposes into the direct sum of the antichiral spinor representation, the vector, and the trivial representation with respect to $\mathfrak{spin}(2,8)$, it follows that the charge matrix of $\frac{2}{5}$ BPS solutions of $\mathcal{N}=5$ supergravity satisfy both $\mathscr{C}^3=0$ and $\mathrm{ad}_{\mathscr{C}}{}^3=0$, but $\mathscr{C}^2\neq 0$. The same property holds then for charge matrices of the $\frac{1}{4}$ BPS solutions of $\mathcal{N}=8$ supergravity and for the elements of $\mathcal{M}_2=\mathcal{M}_{(2,0)}\cup\mathcal{M}_{(1,1)}$ in $\mathcal{N}=6$ supergravity.

One computes that the $\frac{1}{2}$ BPS solutions of $\mathcal{N}=6$ supergravity have charge matrices which satisfy $\mathscr{C}^2=0$, from which it follows that $\mathrm{ad}_{\mathscr{C}}{}^3=0$, and so the $\frac{1}{2}$ BPS solutions of $\mathcal{N}=8$ supergravity have charge matrices which satisfy $\mathscr{C}^3=0$. Note finally that $\mathscr{C}^2=0$ implies $\mathscr{C}=0$ for $\mathfrak{e}_{8(8)}\ominus\mathfrak{spin}^*(16)$, and therefore the nilpotency degree of the charge matrix in the adjoint representation does not disentangle the $\frac{1}{2}$ BPS solutions from the $\frac{1}{4}$ BPS ones. It is useful then to consider the embedding of $\mathcal{N}=4$ supergravity coupled to six vector multiplets inside maximal supergravity. The latter can be understood from the decomposition

$$\mathfrak{e}_{8(8)}\cong\mathfrak{spin}(8,8)\oplus S_+\ . \tag{5.23}$$

Both the $\frac{1}{4}$ and the $\frac{1}{2}$ BPS solutions of $\mathcal{N}=8$ supergravity that are also $\frac{1}{2}$ BPS solutions of $\mathcal{N}=4$ supergravity coupled to six vector multiplets have charge matrices which are nilpotent in the spinor representation ${\mathscr{C}_{S_-}}^2=0$. As it will be explained in the final section, the difference between $\frac{1}{4}$ and $\frac{1}{2}$ BPS solutions of $\mathcal{N}=8$ supergravity is characterised in $\mathcal{N}=4$ supergravity by the fact that the $\mathfrak{spin}(8,8)$ charge matrix corresponding to the latter are also nilpotent in the vector representation $[\mathscr{C}_{S_-},[\mathscr{C}_{S_-},\Gamma^{\mathcal{M}}]]=0$, whereas the $\frac{1}{4}$ BPS ones are not.

In order to characterise this $\frac{1}{4}/\frac{1}{2}$ difference in maximal supergravity, one has to consider (for example) the charge matrix in the **3875** representation of $\mathfrak{e}_{8(8)}$ that arises in the decomposition of the rank two symmetric tensor of the adjoint representation. As well as the adjoint representation, the **3875** is five-graded with respect to the subgroup $\mathrm{SL}(2,\mathbb{R})\times E_{7(7)}$ (see appendix A), therefore the quintic characteristic equation is also valid in the **3875** representation. It follows that the BPS charge matrix satisfies ${\mathscr{C}_{\mathbf{3875}}}^5=0$. The **3875** of $E_{8(8)}$ decomposes into the following representations of Spin(8,8) [47]

$$\mathbf{3875}\cong(V\otimes V)_{\square\!\square}\oplus(S_-\otimes S_-)_{\tiny\begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix}}\oplus(V\otimes S_-)_{\mathbf{1920}} \tag{5.24}$$

The action of $\mathscr{C}$ in the tensor product representation

$$\mathscr{C}_{S_-\otimes S_-}\equiv\mathbb{1}\otimes\mathscr{C}_{S_-}+\mathscr{C}_{S_-}\otimes\mathbb{1} \tag{5.25}$$

to the third power

$${\mathscr{C}_{S_-\otimes S_-}}^3=\mathbb{1}\otimes{\mathscr{C}_{S_-}}^3+3\mathscr{C}_{S_-}\otimes{\mathscr{C}_{S_-}}^2+3{\mathscr{C}_{S_-}}^2\otimes\mathscr{C}_{S_-}+{\mathscr{C}_{S_-}}^3\otimes\mathbb{1} \tag{5.26}$$

vanishes if ${\mathscr{C}_{S_-}}^2=0$. Then if both ${\mathscr{C}_{S_-}}^2=0$ and ${\mathscr{C}_V}^2=0$, it follows in the same way that

$${\mathscr{C}_{\square\!\square}}^3={\mathscr{C}_{\tiny\begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix}}}^3={\mathscr{C}_{\mathbf{1920}}}^3=0\ . \tag{5.27}$$

The charge matrices associated to $\frac{1}{2}$ BPS solutions of $\mathcal{N}=8$ supergravity thus satisfy that

$${\mathscr{C}_{\mathbf{3875}}}^3=3\big(\mathscr{C}\otimes\mathscr{C}^2+\mathscr{C}^2\otimes\mathscr{C}\big)_{\mathbf{3875}\otimes\mathbf{3875}}=0\ . \tag{5.28}$$

However, if ${\mathscr{C}_{S_-}}^2=0$ but ${\mathscr{C}_V}^2\neq0$,

$${\mathscr{C}_{\square\!\square}}^4=6\big({\mathscr{C}_V}^2\otimes{\mathscr{C}_V}^2\big)_{\square\!\square\otimes\square\!\square}\neq0\ . \tag{5.29}$$

Therefore, the charge matrices associated to $\frac{1}{4}$ BPS solutions of $\mathcal{N}=8$ supergravity are such that ${\mathscr{C}_{\mathbf{3875}}}^4\neq0$.

The nilpotency degrees of the extremal Noether charges within $\mathcal{N}=5,6,8$ supergravity are summarised in table 8.

The conjectured additional stratum $\mathcal{M}_{0^\circ}$ is in the same complex orbit of E_8 as $\mathcal{M}_1$, and the corresponding charge matrix thus satisfy the same nilpotency condition. However, the $E_{7(7)}$ invariant is strictly negative in this case. Such non-BPS solutions would correspond to particular values of the conserved charges for which the purely gravitational contribution to horizon area cancels exactly the one associated to central charges of negative $E_{7(7)}$ invariant.

	$\mathcal{N}=5$		$\mathcal{N}=6$		$\mathcal{N}=8$		
$\mathcal{M}_1$	$\mathscr{C}^3=0$	$\mathrm{ad}_{\mathscr{C}}{}^5=0$	$\mathscr{C}^3=0$	$\mathrm{ad}_{\mathscr{C}}{}^5=0$	$\mathscr{C}^5=0$	$\mathscr{C}_{\mathbf{3875}}{}^5=0$	$\lozenge>0$
$\mathcal{M}_{1^\circ}$			$\mathscr{C}^3=0$	$\mathrm{ad}_{\mathscr{C}}{}^4=0$	$\mathscr{C}^4=0$	$\mathscr{C}_{\mathbf{3875}}{}^5=0$	$\lozenge=0$
$\mathcal{M}_2$	$\mathscr{C}^3=0$	$\mathrm{ad}_{\mathscr{C}}{}^3=0$	$\mathscr{C}^3=0$	$\mathrm{ad}_{\mathscr{C}}{}^3=0$	$\mathscr{C}^3=0$	$\mathscr{C}_{\mathbf{3875}}{}^5=0$	$\lozenge=0$
$\mathcal{M}_4$			$\mathscr{C}^2=0$	$\mathrm{ad}_{\mathscr{C}}{}^3=0$	$\mathscr{C}^3=0$	$\mathscr{C}_{\mathbf{3875}}{}^3=0$	$\lozenge=0$

Table 8. Nilpotency degree of the charge matrices for $\mathcal{N}=5$, 6, 8.

Of course these nilpotency conditions also define the corresponding nilpotent orbits in $\mathfrak{N}_{\mathfrak{G}}$. As we have explained in this section, the moduli spaces $\mathcal{M}_n$ are Lagrangian submanifolds of the corresponding orbits in $\mathfrak{N}_{\mathfrak{G}}$, with respect to the symplectic structure associated to the Lie algebra. The link between extremal black hole solutions of maximal supergravity and these nilpotent orbits was already noticed in [39]. It turns out that the representations of $E_{8(8)}$ on the nilpotent orbits of $\mathfrak{N}_{E_{8(8)}}$ lead to unitary representations of $E_{8(8)}$ on the space of functions supported on Lagrangian submanifolds (see [48] for the case of $E_{8(-24)}$). There have been speculations that such "quantised" representations of $E_{8(8)}$ would play a role in the quantisation of black holes [49]. It is rather natural to conjecture that there exist unitary representations of the group $\mathfrak{G}$ on the moduli spaces $\mathcal{M}_n$ which are induced by the adjoint action of $\mathfrak{G}$ on the corresponding nilpotent orbits of $\mathfrak{N}_{\mathfrak{G}}$ in which $\mathcal{M}_n$ can be embedded as Lagrangian submanifolds. The associated $\mathfrak{G}$ symmetric quantum mechanics on the moduli spaces of extremal spherically symmetric black holes might permit one to compute non-perturbative corrections to the action defining the stationary equations of motion of supergravity theories.

5.2 Active duality transformations and parabolic cosets

Unlike the elements of the divisor subgroup $\mathfrak{H}^* \subset \mathfrak{G}$, a general element $g \in \mathfrak{G}$ does not in general preserve asymptotic conditions through the standard non-linear action. Nevertheless, for $d \geq 4$, it is possible to define an action of the whole duality group, different from the standard non-linear action, which preserves asymptotic conditions in such a way that the action on electromagnetic charges is the same as the standard non-linear action [15].

Action of the four-dimensional duality group $\mathfrak{G}_4$. In four dimensions the electromagnetic charges transform in a representation $\mathfrak{l}_4$ of the duality group $\mathfrak{G}_4$. Given any $g \in \mathfrak{G}_4$ and any particular set $Z \in \mathfrak{l}_4$ of such charges, there exists a Borel subgroup $\mathfrak{B}_Z \subset \mathfrak{G}_4$ that leaves Z conformally invariant (that is, invariant up to a factor),

$$g\,Z = \lambda_{(g,Z)}\,Z \qquad \lambda_{(g,Z)} \in \mathbb{R}_+^* \tag{5.30}$$

and which is big enough to act transitively on the symmetric space $\mathfrak{G}_4/\mathfrak{H}_4$. Furthermore, there is a distinguished generator $\boldsymbol{z} \in \mathfrak{B}_Z$ such that any element of $\mathfrak{B}_Z$ decomposes as the product of an element $\exp(\ln\lambda\, \boldsymbol{z})$ and an element that leaves invariant the charge Z, such that $\mathfrak{B}_Z \cong \mathbb{R}_+^* \ltimes \mathfrak{B}_{0\,Z}$. By the Iwasawa theorem, we can represent g in the form

$$g = u_{(g,Z)} \exp\Big(\ln\lambda_{(g,Z)}\, \boldsymbol{z}\Big)\, b_{(g,Z)} \tag{5.31}$$

with $u_{(g,Z)} \in \mathfrak{H}_4$ and $b_{(g,Z)} \in \mathfrak{B}_{0\,Z}$. Of the three factors in (5.31), only the first leaves invariant the asymptotics of the scalar fields. However, due to the invariance of Z under the last factor, we need only worry about implementing the action of the middle (scaling) operator in a way compatible with the asymptotics. This is what the so-called 'trombone symmetry' is needed for.

As originally defined in [15], the trombone symmetry is a symmetry of the equations of motion of any pure supergravity in any dimension, but it is not a symmetry of the action. It acts on the fields as a rescaling of the various tensor fields with a weight given by their rank; on the metric, the vectors and the scalars it thus acts as

$$g_{\mu\nu}(x) \rightarrow \lambda^2\, g_{\mu\nu}(x) \qquad A_\mu(x) \rightarrow \lambda A_\mu(x) \qquad \phi(x) \rightarrow \phi(x)\ . \tag{5.32}$$

In other words, this symmetry acts like a Weyl transformation with a constant parameter λ. By the diffeomorphism invariance of the theory, the above action is equivalent to a coordinate rescaling $\varphi(x) \rightarrow \varphi(\lambda^{-1}x)$ on all fields *without* rescaling the various tensor fields according to their rank. By definition, this compensated trombone transformation preserves the asymptotic behaviour of the solution, and acts on the charge Z by a rescaling, precisely as in (5.30). Consequently, the action of an element $g \in \mathfrak{G}_4$ of the active duality group on a solution with charge Z, is defined, via the Iwasawa decomposition (5.31), as the successive action of the compensated trombone symmetry of parameter $\lambda_{(g,Z)}$ and the standard non-linear action of the element $u_{(g,Z)} \in \mathfrak{H}_4$. By construction, the action of the active duality group preserves the asymptotic behaviour of the solution and acts on the charge Z as in (5.30). However, it does not preserve the number of preserved supersymmetry charges in general. Nevertheless, non-supersymmetric solutions remain non-supersymmetric under the action of the active duality group $\mathfrak{G}_4$. Although (5.30) would seem to suggest that one can take $\lambda \rightarrow 0$, this limit is *not* in the orbit space: the Iwasawa decomposition (5.31) holds for *any* element $g \in \mathfrak{G}_4$ with *non-zero* $\lambda > 0$. In other words, *the group $\mathfrak{G}_4$ does not mix BPS and non-BPS solutions.* As we will see below this is a crucial difference with respect to the action of the three-dimensional duality group $\mathfrak{G}$ whose maximal subgroup $\mathfrak{H}^*$ is non-compact.

From the above discussion, it follows that the $\mathfrak{G}_4$-orbits are of the form

$$\frac{\mathfrak{G}_4}{\mathfrak{B}_{0\,Z}} \cong \mathbb{R}_+^* \times \frac{\mathfrak{H}_4}{\mathfrak{B}_{0\,Z} \cap \mathfrak{H}_4}\ . \tag{5.33}$$

The fact that these orbits take the form of *parabolic cosets* over the group $\mathfrak{G}_4$ explains why we have a proper group action of the full group $\mathfrak{G}_4$ on them. Since the active transformations act on the charges linearly, one can furthermore restrict the action of $\mathfrak{G}_4$ to an arithmetic subgroup that preserves the Dirac quantisation condition and acts linearly on the lattice of quantised charges [15]. For maximal $\mathcal{N} = 8$ supergravity, the parabolic stability groups $\mathfrak{B}_{0\,Z} \subset E_{7(7)}$ of the 56 electromagnetic charges and their $E_{7(7)}$-orbits were analysed and classified in [35].

Action of three-dimensional duality group $\mathfrak{G}$. We now wish to generalise this construction to three dimensions in such a way that an action of the full duality group $\mathfrak{G}$ can be implemented on the orbits. In three dimensions, the charges are associated to the scalar fields themselves, and they transform in the adjoint representation of $\mathfrak{G}$. In the adjoint representation, the subgroup of $\mathfrak{G}$ that leaves a given element of its Lie algebra $\mathfrak{g}$ conformally invariant (the would-be analogue of $\mathfrak{B}_0$) is not big enough to act transitively on $\mathfrak{G}/\mathfrak{H}^*$. However, as we are going to see, one can nevertheless generalise the concept of active transformations to three dimensions. There are several new features and subtleties here, which we will now explain in turn.

From the five-graded decomposition of $\mathfrak{g}$, one can define a maximal parabolic subgroup $\mathfrak{P} \subset \mathfrak{G}$, whose Lie algebra $\mathfrak{p}$ consists of all generators with non-negative gradation, i.e.

$$\mathfrak{p} \cong \mathbf{1}^{(0)} \oplus \mathfrak{g}_4^{(0)} \oplus \mathfrak{l}_4^{(1)} \oplus \mathbf{1}^{(2)} \ . \tag{5.34}$$

The gradation is defined with respect to the generator $\boldsymbol{h} \in \mathfrak{g}$, and $\mathfrak{P} \cong \mathbb{R}_+^* \ltimes \mathfrak{P}_0$ where $\mathfrak{P}_0 \subset \mathfrak{P}$ is the subgroup generated by

$$\mathfrak{p}_0 \cong \mathfrak{g}_4^{(0)} \oplus \mathfrak{l}_4^{(1)} \oplus \mathbf{1}^{(2)} \tag{5.35}$$

from which the generator $\boldsymbol{h}$ has been omitted. The maximal parabolic subgroup $\mathfrak{P}$ can be associated to the charge matrix $\mathscr{C} = c\boldsymbol{h}$ similarly to the way that the Borel subgroup $\mathfrak{B}_Z \subset \mathfrak{G}_4$ can be associated to a given charge Z in higher dimensions (we assume $c > 0$ for the moment). By contrast, the adjoint action of $\mathfrak{P}_0$ does not leave the generator $\boldsymbol{h}$ invariant, but only its subgroup $\mathfrak{G}_4$ does: from the four-dimensional point of view, a solution associated to the charge matrix $\mathscr{C} = c\boldsymbol{h}$ is purely gravitational, while the action of the $\mathfrak{G}_4$ subgroup only shifts the scalar fields by constants.

We use the common convention that the $\mathfrak{G}/\mathfrak{H}^*$ coset representative $\mathcal{V}$ is defined as a function on the parabolic subgroup $\mathfrak{P}$, for which the $\mathfrak{G}_4$ component is defined to be a given representative of a coset element $\mathfrak{G}_4/\mathfrak{H}_4$. Then the action of an element $p \in \mathfrak{P}$ on $\mathcal{V}$ only requires a right compensating transformation $h_4 \in \mathfrak{H}_4 \subset \mathfrak{H}^*$

$$\mathcal{V}(p) = p\,\mathcal{V}\,h_4(p, \mathcal{V}) \,, \tag{5.36}$$

needed to compensate for the component of p lying in $\mathfrak{G}_4$. It follows that the generators of $\mathfrak{l}_4^{(1)}$ act on the electromagnetic scalars by constant shifts. The latter decompose into two subsets. Half of them act on the scalars arising from the time components of the Maxwell one-forms as global gauge transformations[22]

$$A + i\alpha dt = e^{-i\alpha\, t}\big(d + A\big)e^{i\alpha\, t} \ . \tag{5.37}$$

The other half correspond to shifts of the integration constants appearing in the definitions of the scalar fields dual to three-dimensional one-forms associated to the dimensionally

[22] For a non-zero NUT charge, the timelike isometry orbits are compact and there is a topological quantisation condition on the parameter α. However we will interpret the action of these generators as large gauge transformations when acting on a solution with charge matrix $\mathscr{C} = c\boldsymbol{h}$ for which the timelike isometry orbits are non-compact and α can then take arbitrary values.

reduced Maxwell fields. We conclude that the action of the generators of $\mathfrak{l}_4^{(1)}$ on a solution can be interpreted as large gauge transformations. Likewise, the action of the generator $\boldsymbol{e} \in \mathbf{1}^{(2)}$ on a solution amounts to a shift of the integration constant appearing in the definition of the axion field obtained from the four-dimensional metric by dualisation in three dimensions. Therefore, the action of the group $\mathfrak{P}_0$ on a solution with a charge matrix $\mathscr{C} = c\boldsymbol{h}$ amounts to a reparametrisation of the solution. In other words: although the map $\mathcal{V} \to p_0\mathcal{V}$ for $p_0 \in \mathfrak{P}_0$ changes the asymptotics of $\mathcal{V}$, in which case the scalar field configurations $\mathcal{V}$ and $p_0\mathcal{V}$, for $p_0 \in \mathfrak{P}_0$, would be regarded as inequivalent from the point of view of the three-dimensional theory, they are in fact physically equivalent from the point of view of the four-dimensional theory because the constant shifts induced by p_0 all drop out in the relevant charges as computed in four dimensions. The present construction thus retains a 'memory' of the four-dimensional origin of the three-dimensional theory.

The remaining generator of the maximal parabolic subgroup $\mathfrak{P}$ is the generator $\boldsymbol{h}$ itself. It follows from the five-graded decomposition (2.1) of $\mathfrak{g}$ that its action on a given solution is again a trombone-like symmetry. The latter is a modified version of (5.32) which scales spacelike and timelike indices differently, and which *only exists for stationary solutions.* More specifically, we have

$$\begin{aligned} &g_{00}(x) \to \lambda^2 g_{00}(x) \qquad &&g_{0\mu}(x) \to g_{0\mu}(x) \qquad &&g_{\mu\nu}(x) \to \lambda^{-2} g_{\mu\nu}(x) \\ &A_0(x) \to \lambda A_0(x) &&A_\mu(x) \to \lambda^{-1} A_\mu(x) && \end{aligned} \tag{5.38}$$

where x^μ now denotes the *spatial* coordinates, and Greek indices are understood to run from 1 to 3. By diffeomorphism covariance, this action on stationary solutions is equivalent, to the 'compensated trombone' transformation

$$t \to \lambda t \qquad x^\mu \to \lambda^{-1} x^\mu \,, \tag{5.39}$$

i.e., to a 'weighted' rescaling of the four-dimensional coordinates (t, x^μ) without rescaling the tensor fields with respect to their rank.[23]

For any other charge matrix $\mathscr{C}$ in the $\mathfrak{H}^*$-orbit of $c\boldsymbol{h}$ we have $\mathscr{C} = U_{\mathscr{C}}(c\boldsymbol{h})U_{\mathscr{C}}^{-1}$ for some $U_{\mathscr{C}} \in \mathfrak{H}^*$. Consequently we can define the associated maximal parabolic subgroup $\mathfrak{P}_{\mathscr{C}} = U_{\mathscr{C}}\mathfrak{P}U_{\mathscr{C}}^{-1} \subset \mathfrak{G}$ whose Lie algebra $\mathfrak{p}_{\mathscr{C}} \subset \mathfrak{g}$ is generated by the eigenvectors of the adjoint action of $\mathscr{C}$ with positive eigenvalues. As for $\mathfrak{P}$, any element of $\mathfrak{P}_{\mathscr{C}}$ can be written as the product of an element of the form $\exp(c^{-1} \ln \lambda \mathscr{C})$ and an element of the subgroup $\mathfrak{P}_{0\,\mathscr{C}} \subset \mathfrak{P}_{\mathscr{C}}$.

Inspired by the definition of the active duality group transformations in [15], we now define the active transformations in the three-dimensional theory in such a way that the

[23]That (5.38) is indeed correct is most easily seen for pure gravity in four dimensions: using $g_{00} = -H$, $g_{0\mu} = -H\hat{B}_\mu$, the duality relation $H^2 d\hat{B} = \star dB$ and the standard Kaluza Klein formula

$$g_{\mu\nu} = H^{-1}\gamma_{\mu\nu} - H\hat{B}_\mu\hat{B}_\nu \,,$$

we see that the three-dimensional fields scale as $H \to \lambda^2 H$ and $B \to \lambda^2 B$ (as it must be, since (H, B) coordinatise the σ-model manifold $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$), while $\gamma_{\mu\nu}$ is invariant. This corresponds precisely to the action of $\boldsymbol{h}$ in the five-graded decomposition (2.1).

action of an element of the maximal parabolic subgroup $\mathfrak{P}_{\mathscr{C}}$ on a solution of charge matrix $\mathscr{C}$ is given by the compensated trombone transformation with parameter given by the component of the $\mathfrak{P}_{\mathscr{C}}$ element associated to the generator $\mathscr{C}$. However, there is another subtlety which distinguishes the three-dimensional theory from the four-dimensional one, and which is related to the fact that the maximal subgroup $\mathfrak{H}^*$ is not compact, unlike the group $\mathfrak{H}_4$ in (5.31). If we were dealing with the compact form $\mathfrak{H}$ instead (as would be the case for Lorentzian solutions corresponding to the reduction with a spacelike Killing vector), the Iwasawa theorem would entail the isomorphism

$$\frac{\mathfrak{G}}{\mathfrak{P}} \cong \frac{\mathfrak{H}}{\mathfrak{H} \cap \mathfrak{P}} = \frac{\mathfrak{H}}{\mathfrak{H}_4} \tag{5.40}$$

such that the moduli space of charges could be identified with the parabolic coset

$$\mathcal{M}^{\text{Lorentz}} \cong \frac{\mathfrak{G}}{\mathfrak{P}_0}\,, \tag{5.41}$$

in complete analogy with (5.33). The formula (5.40) would furthermore ensure that a proper action of the full group $\mathfrak{G}$ can be implemented on the full orbit space. Here, by contrast, the maximal subgroup $\mathfrak{H}^* \subset \mathfrak{G}$ is *non-compact*. Because the Iwasawa decomposition does not generally hold with maximal non-compact subgroups, the isomorphism (5.40) is no longer valid if we replace $\mathfrak{H}$ by $\mathfrak{H}^*$, so stationary solutions cannot fully be described in terms of parabolic coset spaces. Rather, the breakdown of the Iwasawa theorem is precisely linked to the existence of BPS orbits, whereas the isomorphism (5.41) is possible for spacelike reductions because of the absence of BPS colliding plane wave solutions. Indeed, the following analysis will trace out in detail the link between different types of BPS orbits and the subsets of $\mathfrak{G}$ for which the Iwasawa decomposition fails, and will relate them to the strata $\mathcal{M}_n$ discussed in the foregoing section.

For a non-compact maximal subgroup $\mathfrak{H}^*$, the Iwasawa theorem only holds on a *dense subset* $\mathring{\mathfrak{G}} \subset \mathfrak{G}$. Every element $g \in \mathring{\mathfrak{G}}_{\mathscr{C}} \equiv U_{\mathscr{C}} \mathring{\mathfrak{G}} U_{\mathscr{C}}^{-1}$ in this dense subset can be decomposed into a product of an element $u_{(g,\mathscr{C})} \in \mathfrak{H}^*$, a 'diagonal' element $\exp(c^{-1} \ln \lambda \mathscr{C})$ (with $\lambda_{g,\mathscr{C}} > 0$) and an element $p \in \mathfrak{P}_{0\mathscr{C}}$ as follows

$$g = u_{(g,\mathscr{C})} \exp\left(c^{-1} \ln \lambda_{(g,\mathscr{C})} \mathscr{C}\right) p_{(g,\mathscr{C})}\ . \tag{5.42}$$

The singular elements $g \in \mathfrak{G} \setminus \mathring{\mathfrak{G}}_{\mathscr{C}}$ (where the Iwasawa decomposition breaks down) correspond to limits of regular elements $g_k \in \mathring{\mathfrak{G}}_{\mathscr{C}}$ for which $\lambda_{(g_k,\mathscr{C})} \to 0$, while simultaneously the element $u_{(g_k,\mathscr{C})}$ goes to the boundary of the non-compact group $\mathfrak{H}^*$, in such a way that the limit $g = \lim g_k \in \mathfrak{G}$ is well-defined. This is one main difference with (5.31) for which no such limit can be taken because $\mathfrak{H}_4$ is compact.

The active duality group transformation corresponding to an element $g \in \mathring{\mathfrak{G}}_{\mathscr{C}}$ on a solution $\mathcal{V}(x)$ with a charge matrix $\mathscr{C}$ with $c > 0$ is now defined as the successive action of the compensated trombone transformation with parameter $\lambda_{(g,\mathscr{C})}$, followed by the standard non-linear action of the group element $u_{(g,\mathscr{C})} \in \mathfrak{H}^*$ [as computed from (5.42); note that this decomposition depends on the initial solution $\mathcal{V}$ via its associated charge $\mathscr{C}$], i.e.

$$g\,:\,\mathcal{V}(x)\,\to\,\mathcal{V}'(x) := u_{(g,\mathscr{C})} \cdot \mathcal{V}\big(\lambda^{-1}_{(g,\mathscr{C})} x\big) \cdot h\big(u_{(g,\mathscr{C})}, \mathcal{V}(\lambda^{-1}_{(g,\mathscr{C})} x)\big)\,, \tag{5.43}$$

where the matrix $\mathcal{V}(x)$ is triangular (i.e. $\mathcal{V} \in \mathfrak{P}_{\mathscr{C}}$), and the compensator $h \in \mathfrak{H}^*$ restores the triangular gauge, but now with respect to $\mathfrak{P}_{\mathscr{C}(g)}$, where the transformed charge matrix is computed from (2.9) as

$$\mathscr{C}(g) = \lambda_{(g,\mathscr{C})}\, u_{(g,\mathscr{C})}\, \mathscr{C}\, u^{-1}_{(g,\mathscr{C})} \tag{5.44}$$

while the BPS parameter transforms as

$$c(g) = \lambda_{(g,\mathscr{C})}\, c\ . \tag{5.45}$$

The remarkable fact is now that these transformations define regular (and non-trivial!) solutions even when $u_{(g,\mathscr{C})}$ and $\lambda_{(g,\mathscr{C})}$ become singular separately. For $\lambda_{(g_k,\mathscr{C})} \to 0$ we have $\lim c(g_k) = 0$, and therefore the initial non-BPS solution is mapped to a BPS solution. From (5.42), we see that the limiting matrix $g = \lim_k g_k$ no longer admits an Iwasawa decomposition with respect to $\mathscr{C}$. Consequently, *the elements $g \in \mathfrak{G}$ for which the Iwasawa decomposition fails are precisely the ones that map non-BPS to BPS solutions.* However, as we already indicated, this procedure fails to define a proper Lie group action in general owing to the existence of non-trivial solutions with $c = 0$. As defined above the action of the active duality group cannot be 'inverted' in the sense that the above procedure cannot be applied to solutions with vanishing BPS parameter, because there are generators in the Lie algebra $\mathfrak{g}$ whose action diverges in the limit $c \to 0$. In other words, the group $\mathfrak{G}$ cannot act properly on all solutions.

One can understand the 'almost action' of the active duality group from a more geometrical point of view. The 'almost Iwasawa decomposition' (5.42) permits one to define[24] a homeomorphism between $\mathcal{M}_0 \cong \mathbb{R}^*_+ \times \mathfrak{H}^*/\mathfrak{H}_4$ and $\mathring{\mathfrak{G}}/\mathfrak{P}_0$. The 'almost action' of $\mathring{\mathfrak{G}}$ on $\mathcal{M}_0$ can then be derived from the action of $\mathfrak{G}$ on $\mathring{\mathfrak{G}}/\mathfrak{P}_0$ using this homeomorphism. One cannot extend this 'almost action' to a Lie group action on $\mathcal{M}$ because the codimension of $\mathcal{M}_1$ in $\mathcal{M}_0$ (which equals 1) does not match the codimension of the subset $\mathfrak{G} \setminus \mathring{\mathfrak{G}}$ on which the Iwasawa decomposition fails. More specifically, the homeomorphism between $\mathcal{M}_0$ and $\mathring{\mathfrak{G}}/\mathfrak{P}_0$ does not extend to a homeomorphism between $\mathcal{M}$ and $\mathfrak{G}/\mathfrak{P}_0$. Basically, the dimension of the complement of $\mathcal{M}_0$ inside $\mathfrak{G}/\mathfrak{P}_0$ is of lower dimension than the next stratum $\mathcal{M}_1 \cong \mathfrak{H}^*/\mathfrak{I}_1$, in such a way that the moduli space of charges $\mathcal{M}$ cannot be homeomorphic to the coset space $\mathfrak{G}/\mathfrak{P}_0$. When $\mathfrak{H}^*$ admits a U(1) factor,

$$\mathcal{M}_0 \cong \mathbb{R}^*_+ \times \mathfrak{H}^*/\mathfrak{H}_4 \cong \frac{\mathbb{C}^\times \times \mathfrak{H}_4 \ltimes \mathfrak{l}_4}{\mathfrak{H}_4} \tag{5.46}$$

and $\mathcal{M}_0$ is locally isomorphic to $\mathbb{C}^\times \times \mathfrak{l}_4$. The complement of the image of the embedding of $\mathcal{M}_0 \cong \mathbb{R}^*_+ \times \mathfrak{H}^*/\mathfrak{H}_4$ into $\mathfrak{G}/\mathfrak{P}_0$ inside $\mathfrak{G}/\mathfrak{P}_0$ corresponds to limit points of $\mathbb{C}^\times \times \mathfrak{l}_4$ for which the complex parameter goes to zero as the vector of $\mathfrak{l}_4$ diverges. It follows that this subspace has same the dimension as $\mathfrak{l}_4$, whereas the stratum $\mathcal{M}_1$ is of dimension $\dim[\mathfrak{l}_4]+1$.

$$\dim\bigl[\mathcal{M}_1\bigr] = \dim\left[\frac{\mathfrak{G}}{\mathfrak{P}_0} \setminus \mathcal{M}_0\right] + 1 \quad \Rightarrow \quad \frac{\mathfrak{G}}{\mathfrak{P}_0} \not\cong \mathcal{M} \cong \mathcal{M}_0 \cup \mathcal{M}_1 \cup \cdots\ . \tag{5.47}$$

[24]Note that the action of $\mathfrak{P}_0$ is well-defined on the submanifold $\mathring{\mathfrak{G}} \subset \mathfrak{G}$, since, by definition, its action on $\mathfrak{G}$ preserves the property of admitting an Iwasawa decomposition.

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Note, however, that the above argument works only for $\mathcal{N} \leq 5$; for $\mathcal{N} = 8$ one would need to better understand how to characterise the subsets of $E_{8(8)}$ on which the Iwasawa decomposition fails. These conclusions can also be stated differently as follows: while there exists an 'almost action' of $\mathfrak{G}$ on the main stratum $\mathcal{M}_0$, no proper action of $\mathfrak{G}$ can be implemented on the various BPS strata: these being Lagrangian submanifolds, they have only half the dimension that would be required for a non-linear realisation of $\mathfrak{G}$.

In this discussion, we have not really been able to precisely generalise the notion of active duality-group transformations to the three-dimensional theories. In this connection, one can identify two noteworthy differences with respect to the higher-dimensional cases which seem to be unavoidable. First, the action of the active duality group on the relevant charges is no longer equivalent to the standard non-linear action of the group. Second, this action is highly non-linear, which follows from the fact that the charge matrix involves gravity degrees of freedom as well. We conclude that the common idea that the three-dimensional duality group $\mathfrak{G}$ is broken at the quantum level to an arithmetic subgroup, with the relevant representation simply defined over the integers, might be too naïve.

Nevertheless, the difficulties that appear in trying to define a non-linear realisation of an arithmetic group, could as well give a solution to the singular behaviour of the 'almost representation' on the BPS strata. Our expectation is that even if there is no well-defined action of $\mathfrak{G}$ on the moduli space $\mathcal{M}$, the space of functions on $\mathcal{M}$ could admit a non-linear action of the duality group $\mathfrak{G}$. We have already seen in the last section that the strata of $\mathcal{M}\backslash\mathcal{M}_0$ are Lagrangian submanifolds of the corresponding nilpotent orbits in $\mathfrak{N}_{\mathfrak{G}}$. $\mathfrak{H}^*/\mathfrak{H}_4$ is itself a Lagrangian submanifold of the $\mathfrak{G}/\mathfrak{G}_4$-orbit of the generator $\boldsymbol{h} \in \mathfrak{g}$. It seems possible that the action of $\mathfrak{G}$ on a solution in $\mathfrak{g}$ of the characteristic equation (2.21, 2.22) induces an action of $\mathfrak{G}$ on the space of functions defined on $\mathcal{M}$. For instance, the stratum $\mathcal{M}_4$ of $\frac{1}{2}$ BPS solutions of $\mathcal{N} = 8$ supergravity is a 29 dimensional Lagrangian submanifold of the minimal adjoint orbit $\mathfrak{G}/\mathfrak{P}_0$, and the minimal unitary realisation of $E_{8(8)}$ [20, 21] might be defined on the functions supported on $\mathcal{M}_4$. The non-perturbative corrections to the three-dimensional Euclidean theory describing stationary solutions of $\mathcal{N} = 8$ supergravity should be invariant under the action of an arithmetic subgroup $E_{8(8)}(\mathbb{Z})$ of $E_{8(8)}$. The corresponding automorphic forms can be written [20]

$$\mathscr{E}^{E_{8(8)}}(\Psi) = \big\langle \Psi_{E_{8(8)}(\mathbb{Z})},\, \rho(\mathcal{V})\, \Psi_{\mathrm{Spin}^*(16)} \big\rangle \,, \tag{5.48}$$

where $\Psi_{\mathrm{Spin}^*(16)}$ is the so-called spherical vector, which would in this case be a $\mathrm{Spin}^*(16)$ invariant function over $\mathcal{M}_4$. $\rho(\mathcal{V})$ is the coset element $\mathcal{V}$ in the minimal unitary representation, and $\Psi_{E_{8(8)}(\mathbb{Z})}$ is an $E_{8(8)}(\mathbb{Z})$ invariant distribution defined over $\mathcal{M}_4$. A spherical vector $\Psi_{\mathrm{Spin}^*(16)}$ and its p-adic equivalent defining $\Psi_{E_{8(8)}(\mathbb{Z})}$ have been computed in [20] and [50] respectively. This formula suggests that non-perturbative corrections can be identified as observables of the quantum mechanics of a particle living on $\mathcal{M}_4$ associated to the operator $\rho(\mathcal{V})$.

We are next going to illustrate the definitions of this section with the two simplest examples, namely pure gravity and Maxwell–Einstein theory.

5.3 The $\mathrm{SL}(2,\mathbb{R})$-orbit of Taub–NUT solutions

The simplest example is pure gravity in four dimensions, for which we can define an 'active' realisation of the Ehlers group $\mathrm{SL}(2,\mathbb{R})$ on stationary solutions following the steps described in the foregoing section. The $\mathfrak{sl}(2,\mathbb{R})$ generators of the Ehlers group decompose as

$$h\boldsymbol{h} + e\boldsymbol{e} + \beta\boldsymbol{\beta} = \begin{pmatrix} h & e+\beta \\ -\beta & -h \end{pmatrix} . \tag{5.49}$$

Here, the $\mathrm{SO}(2)$ generator $\boldsymbol{\beta} \equiv \boldsymbol{e} - \boldsymbol{f}$ preserves the asymptotics, while $\boldsymbol{e}$ is the nilpotent generator of the subgroup $\mathfrak{P}_0 \cong \mathbb{R}$. The Iwasawa decomposition of an $\mathrm{SL}(2,\mathbb{R})$ matrix[25] implies that an element of the coset $\mathrm{SL}(2,\mathbb{R})/\mathbb{R}$ decomposes as the product of an $\mathrm{SO}(2)$ element and an element of the parabolic subgroup $\mathfrak{P} \cong IGL_+(1,\mathbb{R})$ as follows

$$\begin{pmatrix} \mu & 0 \\ \mu b & \mu^{-1} \end{pmatrix} = \frac{1}{\sqrt{1+b^2}} \begin{pmatrix} 1 & -b \\ b & 1 \end{pmatrix} \begin{pmatrix} \sqrt{1+b^2}\mu & 0 \\ 0 & \frac{1}{\sqrt{1+b^2}\mu} \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{(1+b^2)\mu^2} \\ 0 & 1 \end{pmatrix} . \tag{5.50}$$

The charge matrix is

$$\mathscr{C} \equiv \begin{pmatrix} m & n \\ n & -m \end{pmatrix} \quad \in \ \mathfrak{sl}(2,\mathbb{R}) \ominus \mathfrak{so}(2) \ . \tag{5.51}$$

Following the steps of the preceding section (in particular formulas (5.44) and (5.45)), the active action of an element of $\mathrm{SL}(2,\mathbb{R})$ on the Schwarzschild solution of unit mass and vanishing NUT parameter is such that the upper triangular matrix on the right in (5.50) can be disregarded, while the diagonal element raises the BPS parameter from $c = m = 1$ to $c = \sqrt{m^2+n^2} = \sqrt{1+b^2}\mu$ through the action of the trombone transformation. Finally, the $\mathrm{SO}(2)$ element determines the value of the mass and the NUT charge in such a way that the new solution has mass $m = \frac{1-b^2}{\sqrt{1+b^2}}\mu$ and NUT charge $n = \frac{2b}{\sqrt{1+b^2}}\mu$. This defines the isomorphism

$$\mathcal{M}_0 \cong \frac{\mathrm{SL}(2,\mathbb{R})}{\mathbb{R}} \tag{5.52}$$

between the moduli space $(m,n) \neq (0,0)$ of Taub–NUT solutions, and the parabolic coset $\mathrm{SL}(2,\mathbb{R})/\mathbb{R}$. The triangular form of the coset element defines only local coordinates on this space. The coset space is a (trivial) line bundle of fibre $\mathbb{R}_+^*$ over the parabolic coset $\mathrm{SL}(2,\mathbb{R})/IGL_+(1,\mathbb{R}) \cong S^1$, and thus is diffeomorphic to a cylinder. This cylinder is covered by the coordinates $(\mu,b) \in \mathbb{R}_+^* \times \mathbb{R}$ plus an $\mathbb{R}_+^*$ half-line defined by the limit $b \to \pm\infty$ and $\mu \to 0$ in such a way that $|b|\mu$ is a finite positive number. The map $\mu' = \mu|b|$, $b' = -\frac{1}{b}$ defines a complementary open set of coordinates for which the limit point coordinates are now regular. This limit point corresponds to the Schwarzschild solution with negative mass $-\mu|b|$. The cylinder is closed at one end by adding the trivial stratum $\mathcal{M}_1$ consisting only of the point $(m,n) = (0,0)$.

[25] Because $\mathrm{SO}^*(2) = \mathrm{SO}(2)$ is compact, a breakdown of the Iwasawa decomposition is not an issue here, which is consistent with the fact that pure gravity in four dimensions does not admit BPS solutions.

In this way one obtains an action of $\mathrm{SL}(2,\mathbb{R})$ on the Taub–NUT solutions defined from the left action on $\mathrm{SL}(2,\mathbb{R})/\mathbb{R}$ through the map $(m,n) = \left(\frac{1-b^2}{\sqrt{1+b^2}}\mu, \frac{2b}{\sqrt{1+b^2}}\mu\right)$. This map has as inverse

$$\mu = \frac{1}{2}\sqrt{(c+m)^2+n^2} \qquad b = \frac{2nc}{(c+m)^2+n^2} \,. \tag{5.53}$$

For a general element of $\mathrm{SL}(2,\mathbb{R})$, $g \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, with $\alpha\delta-\beta\gamma=1$, one obtains the following transformation of the solution's charges:

$$\begin{aligned} m' &= \frac{(\alpha^2-\gamma^2+\beta^2-\delta^2)c+(\alpha^2-\gamma^2-\beta^2+\delta^2)m+2(\alpha\beta-\gamma\delta)n}{\sqrt{2(\alpha^2+\gamma^2+\beta^2+\delta^2)+2(\alpha^2+\gamma^2-\beta^2-\delta^2)\frac{m}{c}+4(\alpha\beta+\gamma\delta)\frac{n}{c}}} \\ n' &= \frac{2(\alpha\gamma+\beta\delta)c+2(\alpha\gamma-\beta\delta)m+2(\alpha\delta+\beta\gamma)n}{\sqrt{2(\alpha^2+\gamma^2+\beta^2+\delta^2)+2(\alpha^2+\gamma^2-\beta^2-\delta^2)\frac{m}{c}+4(\alpha\beta+\gamma\delta)\frac{n}{c}}} \,. \end{aligned} \tag{5.54}$$

To derive these formulas, one first expresses (μ,b) via (m,n), then works out the non-linear action of $\mathrm{SL}(2,\mathbb{R})$ in order to obtain

$$\mu' = (\alpha+\beta b)\mu \quad , \qquad b' = \frac{\gamma+\delta b}{\alpha+\beta b} \tag{5.55}$$

and finally expresses (m',n') in terms of the new parameters (μ',b') as functions of (m,n). This construction extends trivially to non-vanishing angular momentum by taking a Kerr solution as the reference solution. The action is the same with the value of (a/c) kept fixed.

In order to see explicitly that the 'active action' (5.54) is actually the same as the abstract formula (5.44), we must perform an Iwasawa decomposition of the general $\mathrm{SL}(2,\mathbb{R})$ element g, but with $\mathscr{C}$ from (5.51) rather than $\boldsymbol{h}$ as the diagonal element, as in (5.42). After some algebra we arrive at

$$\begin{aligned} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \frac{1}{\sqrt{1+b^2}}\begin{pmatrix} 1 & -b \\ b & 1 \end{pmatrix} \frac{1}{2c}\begin{pmatrix} (c+m)\lambda+(c-m)\lambda^{-1} & n(\lambda-\lambda^{-1}) \\ n(\lambda-\lambda^{-1}) & (c-m)\lambda+(c+m)\lambda^{-1} \end{pmatrix} \\ \times\frac{1}{2c}\begin{pmatrix} 2c-ne & (c+m)e \\ (-c+m)e & 2c+ne \end{pmatrix} \end{aligned} \tag{5.56}$$

where the matrix in the middle is just $\exp\left[c^{-1}(\ln\lambda)\mathscr{C}\right]$, the matrix on the left is the $\mathrm{SO}(2)$ rotation $u_{(g,\mathscr{C})}$, and the matrix to the right is the parabolic element $p_{(g,\mathscr{C})}$ that leaves invariant the charge matrix $\mathscr{C}$ from (5.51) through the active action (5.44). The parameters in these matrices are given by

$$\begin{aligned} b &= \frac{(\gamma-\beta)c+(\gamma+\beta)m+(\delta-\alpha)n}{(\alpha+\delta)c+(\alpha-\delta)m+(\beta+\gamma)n} \qquad \frac{1}{\sqrt{1+b^2}} = \frac{(\alpha+\delta)c+(\alpha-\delta)m+(\beta+\gamma)n}{2c\lambda} \\ \lambda &= \frac{1}{\sqrt{2c}}\sqrt{(\alpha^2+\gamma^2+\beta^2+\delta^2)c+(\alpha^2+\gamma^2-\beta^2-\delta^2)m+2(\alpha\beta+\gamma\delta)n} \\ e &= \frac{n\beta-(c-m)\alpha+\lambda^{-2}\big(n\gamma+(c-m)\delta\big)}{(c-m)\beta+n\alpha} \,. \end{aligned} \tag{5.57}$$

Finally, we would like to point out that it is by no means evident from (5.54) whether and how the continuous duality group $\mathrm{SL}(2,\mathbb{R})$ can be broken to an arithmetic subgroup such as $\mathrm{SL}(2,\mathbb{Z})$ upon quantisation. Although one can of course restrict the action (5.54) to elements g in such an arithmetic subgroup, the resulting (discrete) set of admissible charges (m,n) does not appear to have a nice structure satisfying a Dirac quantisation condition. As this is the simplest example involving gravitational degrees of freedom, similar comments apply to the larger duality groups of all supergravity theories with $\mathcal{N} \geq 1$.

5.4 Maxwell–Einstein theory [$\mathfrak{G} = \mathrm{SU}(2,1)$]

The simplest example including nontrivial BPS solutions (which do not exist for pure gravity) [26] in which one can make completely explicit the failure of the construction to define a group action for a non-compact divisor group $\mathfrak{H}^*$ is Maxwell–Einstein theory, for which the coset space is $\mathrm{SU}(2,1)/\mathrm{U}(1,1)$. Defining a group action of the duality group $\mathfrak{G}$ on the space of solutions requires the vector fields defining the Lie algebra $\mathfrak{g}$ to be regular. If the divisor group $\mathfrak{H}^*$ is non-compact, these vector fields are regular only on a dense subspace of the space of solutions, but diverge like $\frac{1}{c}$ as one approaches the subspace of BPS solutions. For this reason, the action of the duality group will be ill defined on this subspace so that some of the directions in the group degenerate and do not define transformations. Nevertheless, the vector fields do allow for transformations that allow one to move from any non-BPS solution to any other solution with the same angular momentum ratio (c/a), including the BPS solutions (with this fixed ratio).

The Lie algebra $\mathfrak{su}(2,1)$ decomposes into a direct sum of $\mathfrak{su}(1,1)$ and the parabolic subalgebra $\mathfrak{p}$ generated by $\boldsymbol{h}$, $\boldsymbol{\beta}$, $\boldsymbol{e}$, $\boldsymbol{x}$ and $\boldsymbol{y}$ (with the corresponding parameters h, β, e, x and y, respectively). Hence, any element $\boldsymbol{u} \in \mathfrak{su}(2,1)$ has the form

$$\boldsymbol{u} = \begin{pmatrix} i\alpha & \alpha & \xi + i\zeta \\ -\alpha & i\alpha & -\zeta + i\xi \\ \xi - i\zeta & -\zeta - i\xi & -2i\alpha \end{pmatrix} + \begin{pmatrix} h + i\beta & e & x + iy \\ 0 & -h + i\beta & 0 \\ 0 & -y - ix & -2i\beta \end{pmatrix} \tag{5.58}$$

where the left summand is in $\mathfrak{su}(1,1)$. The charge matrix is

$$\mathscr{C} \equiv \begin{pmatrix} m & n & -\frac{z}{\sqrt{2}} \\ n & -m & i\frac{z}{\sqrt{2}} \\ \frac{\bar{z}}{\sqrt{2}} & i\frac{\bar{z}}{\sqrt{2}} & 0 \end{pmatrix} \in \mathfrak{su}(2,1) \ominus \mathfrak{u}(1,1) \; . \tag{5.59}$$

As explained in section 5.2, $\boldsymbol{h}$ acts like the trombone transformation, up to a pseudo-conformal diffeomorphism. The action of $\boldsymbol{e}$ amounts to the addition of an irrelevant constant to the axion field, and $\boldsymbol{y}$ defines a shift of the magnetic scalar in a similar way. The final generator $\boldsymbol{x}$ acts as a global gauge transformation. We thus define the 'active' $\mathrm{SU}(2,1)$ on the space of solutions in such a way that the generators $\boldsymbol{e}$, $\boldsymbol{x}$ and $\boldsymbol{y}$ act trivially on the Kerr solution, and the generator $\boldsymbol{h}$ is defined to act as the compensated trombone transformation on it.

[26] Related results for Maxwell–Einstein theory have been obtained by L. Houart, A. Kleinschmidt, N. Tabti and J. Lindman-Hörnlund (A. Kleinschmidt, priv. comm.).

The generator $\boldsymbol{\beta}$ of the four-dimensional duality group leaves the Kerr solution invariant as it does any pure gravity solution. The isotropy subgroup of SU$(2,1)$ of the Kerr solution under the active transformations is thus the group $\mathfrak{P}_0 \cong I\!c\mathrm{U}(1)$ generated by the Lie algebra elements $\boldsymbol{\beta}$, $\boldsymbol{e}$, $\boldsymbol{x}$ and $\boldsymbol{y}$, which together with the $\boldsymbol{h}$ generator define a maximal parabolic subgroup $\mathfrak{P} \cong \mathbb{R}_+^* \ltimes I\!c\mathrm{U}(1) \subset \mathrm{SU}(2,1)$. The naïve model for the full space of solutions at fixed angular momentum is thus the coset space $\mathrm{SU}(2,1)/I\!c\mathrm{U}(1)$. However, the map from the space of solutions into this coset space fails to be an isomorphism on the subspace of BPS solutions.

The subsequent analysis proceeds along the same lines as for pure gravity. Let us first look to the coset space itself. It is a trivial fibre bundle over $\mathrm{SU}(2,1)/\mathfrak{P} \cong S^3$ with fibre $\mathbb{R}_+^*$. Its (lower) triangular matrix form is

$$\begin{pmatrix} \mu & 0 & 0 \\ \mu(b+i|q|^2) & \mu^{-1} & \sqrt{2}iq \\ \sqrt{2}\mu q^* & 0 & 1 \end{pmatrix} \tag{5.60}$$

with local coordinates $\mu > 0$, $b \in \mathbb{R}$ and $q \in \mathbb{C}$. This coordinate system does not cover the whole coset space. b and q can be regarded as stereographic coordinates on the three-sphere S^3, such that the map

$$\mu' = \sqrt{b^2+|q|^4}\,\mu \qquad b' = -\frac{b}{b^2+|q|^4} \qquad q' = \frac{q}{b-i|q|^2} \tag{5.61}$$

gives the coordinates on the other hemisphere. It remains for one to add the points at infinite b and q with a finite strictly positive value of $\sqrt{b^2+|q|^4}\,\mu$. As for the pure gravity case, such points correspond to Kerr solutions with a negative mass $-\sqrt{b^2+|q|^4}\,\mu$.

The coset matrix (5.60) admits a 'singular Iwasawa decomposition' as a product of an element of $\mathrm{U}(1,1)$, an element generated by $\boldsymbol{h}$ and an element of $I\!c\mathrm{U}(1)$, viz.

$$\begin{pmatrix} \frac{1}{\sqrt{(1-|q|^2)^2+b^2}} & \frac{-b-i|q|^2}{\sqrt{(1-|q|^2)^2+b^2}} & \frac{\sqrt{2}q}{1-|q|^2+ib} \\ \frac{b+i|q|^2}{\sqrt{(1-|q|^2)^2+b^2}} & \frac{1}{\sqrt{(1-|q|^2)^2+b^2}} & \frac{\sqrt{2}iq}{1-|q|^2+ib} \\ \frac{\sqrt{2}q^*}{\sqrt{(1-|q|^2)^2+b^2}} & \frac{-i\sqrt{2}q^*}{\sqrt{(1-|q|^2)^2+b^2}} & \frac{1+|q|^2+ib}{1-|q|^2+ib} \end{pmatrix} \times$$
$$\begin{pmatrix} \sqrt{(1-|q|^2)^2+b^2}\,\mu & 0 & 0 \\ 0 & \frac{1}{\sqrt{(1-|q|^2)^2+b^2}\,\mu} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{b-i|q|^2}{((1-|q|^2)^2+b^2)\mu^2} & \frac{-\sqrt{2}q}{(1-|q|^2+ib)\mu} \\ 0 & 1 & 0 \\ 0 & \frac{i\sqrt{2}q^*}{(1-|q|^2-ib)\mu} & 1 \end{pmatrix}. \tag{5.62}$$

We see that the decomposition becomes singular for the subspace of unit modulus $|q|^2 = 1$ and zero b; that is, the subset of $\mathrm{SU}(2,1)$ on which the Iwasawa decomposition fails is always conjugate to a cylinder $\mathbb{R}_+^* \times S^1$ in $\mathrm{SU}(2,1)$.[27] Nevertheless, one can associate a solution of the Maxwell–Einstein equations to any generic point. If we apply this coset element to the Schwarzschild solution of unit mass (i.e. $m=1, n=z=0$ in (5.59)), the subgroup $\mathfrak{P}_0$ does not act, while the diagonal element changes the BPS parameter from $c \equiv \sqrt{m^2+n^2-|z|^2} = 1$ to $\lambda \equiv \sqrt{(1-|q|^2)^2+b^2}\,\mu$, and the $\mathrm{U}(1,1)$ element yields the transformed charges as

$$
\begin{aligned}
m &= \frac{1-|q|^4-b^2}{\sqrt{(1-|q|^2)^2+b^2}}\,\mu \qquad\qquad n = \frac{2b}{\sqrt{(1-|q|^2)^2+b^2}}\,\mu \\
z &= \frac{1-|q|^2+ib}{\sqrt{(1-|q|^2)^2+b^2}}\,2q\mu\ .
\end{aligned}
\tag{5.63}
$$

Inverting this map we obtain

$$
\begin{aligned}
\mu &= \frac{1}{2}\sqrt{(c+m)^2+n^2} \\
b &= \frac{2nc}{(c+m)^2+n^2} \qquad\qquad q = \frac{z}{c+m+in}\ .
\end{aligned}
\tag{5.64}
$$

In the BPS limit, the map projects out the overall phase of z and $m+in$, which corresponds to the action of the $\mathrm{U}(1)$ center of $\mathrm{U}(1,1)$ on these solutions (this $\mathrm{U}(1)$ rotates $m+in$ and z in the same way, but is not 'seen' by the coordinates (μ,b,q)). With these formulas at hand, we can explicitly verify our previous claim that the combined 'active action' of the two left matrices in (5.62) according to (5.44) and (5.45) remains well-defined even though the matrices separately become singular.

The action of a Lie algebra element of $\mathfrak{su}(2,1)$

$$
\begin{pmatrix}
i\alpha+h & b+\beta & \sqrt{2}(x+iy+r+is) \\
b-\beta & i\alpha-h & \sqrt{2}(y-ix-s+ir) \\
\sqrt{2}(-x+iy+r-is) & -\sqrt{2}(y+ix+s+ir) & -2i\alpha
\end{pmatrix}
\tag{5.65}
$$

is obtained in complete analogy with (5.54). A slightly tedious calculation yields the following infinitesimal action on the elements of the charge matrix:

$$
\begin{aligned}
\delta m &= h\Big(c+\frac{n^2}{c}\Big) + x\frac{pn}{c} + b\frac{mn}{c} + y\frac{qn}{c} + rq + sp + \beta n \\
\delta n &= -h\frac{mn}{c} - x\frac{pm}{c} - b\Big(c+\frac{m^2}{c}\Big) - y\frac{qm}{c} - \beta m + rp - sq \\
\delta q &= h\frac{np}{c} - x\Big(c-\frac{p^2}{c}\Big) + b\frac{mp}{c} - y\frac{qp}{c} - \alpha p + rm - sn \\
\delta p &= -h\frac{nq}{c} - x\frac{qp}{c} - b\frac{mq}{c} + y\Big(c-\frac{q^2}{c}\Big) + \alpha q + sm + qn \\
\delta c &= hm + xq - bn - yp
\end{aligned}
\tag{5.66}
$$

[27] Which also corroborates our previous claim that the set on which the Iwasawa decomposition fails is of codimension 2 in $\mathfrak{G} = \mathrm{SU}(2,1)$.

with $z \equiv q + ip$. This transformation is singular for $c = 0$. Specialising to BPS solutions with $n = p = 0$ and $q, m \neq 0$, we get the action of the four generators of $\mathfrak{su}(2,1) \ominus \mathfrak{u}(1,1)$

$$\begin{aligned}
\delta m &= 0 \\
\delta n &= -(b+y)\frac{m^2}{c} \\
\delta q &= 0 \\
\delta p &= -(b+y)\frac{m^2}{c} \\
\delta c &= (h+x)m\ .
\end{aligned} \tag{5.67}$$

The two generators corresponding to $b + y = h + x = 0$ leave the charges invariant, the one corresponding to $h + x \neq 0$ breaks the BPS condition, and the one corresponding to $b + y \neq 0$ is singular. In fact, this is not the only pathology of the construction. Indeed, if one can reach the BPS Reissner–Nordström solution from any non-BPS one through an action generated by the $\boldsymbol{h}$ generator, one can also reach it from any Kerr–Newman solution with an arbitrary value of the angular momentum per unit of mass. The generalisation to arbitrary angular momentum is trivially obtained by substituting the Kerr solution for the Schwarzschild solution as the starting reference solution on which the maximal parabolic subgroup is defined to act as the trombone symmetry. The orbits are then exactly the same, each with its own value of (a/c). When one reaches a BPS solution, we have $a, c \to 0$ in such a way that this ratio is kept fixed. However, it is not possible to invert this transformation in the sense that there is no preferred value (a/c) from which to start when $a = c = 0$. We conclude that the action of the generators of $\mathfrak{su}(2,1) \ominus \mathfrak{u}(1,1)$ on the BPS solutions is either trivial or ill-defined.

Let us see, anyway, how one can reach BPS solutions from non-BPS solutions through the active action of SU(2,1). For a global transformation $\exp(\ln \lambda \boldsymbol{h})$, one gets

$$\begin{aligned}
m(\lambda) &= \lambda \frac{\sqrt{(c+m)^2+n^2}\, m + (1-\lambda^{-4})\frac{\frac{|z|^2}{4c}+cn^2}{\sqrt{(c+m)^2+n^2}}}{\sqrt{\left(c+m+(1-\lambda^{-2})\frac{|z|^2}{2c}\right)^2+\lambda^{-4}n^2}} \\
n(\lambda) &= \lambda^{-1} n \sqrt{\frac{(c+m)^2+n^2}{\left(c+m+(1-\lambda^{-2})\frac{|z|^2}{2c}\right)^2+\lambda^{-4}n^2}} \\
z(\lambda) &= z \sqrt{\frac{(c+m)^2+n^2}{\left(c+m+(1-\lambda^{-2})\frac{|z|^2}{2c}\right)^2+\lambda^{-4}n^2}} \left(1+(1-\lambda^{-2})\frac{\frac{|z|^2}{2c}-in}{c+m+in}\right)
\end{aligned} \tag{5.68}$$

The BPS parameter is given by

$$c(\lambda) = \lambda c \sqrt{\frac{\left(c+m+(1-\lambda^{-2})\frac{|z|^2}{2c}\right)^2+\lambda^{-4}n^2}{(c+m)^2+n^2}}\ . \tag{5.69}$$

The discriminant for the equation $c(\lambda) = 0$ is strictly negative for non-zero NUT charge n, $\Delta = -\frac{(m+c)^4 n^2}{c^2} + \mathcal{O}(n^4)$. One thus obtains that λ can be chosen in such a way that $c(\lambda) = 0$

if and only if $n = 0$. In the latter case, both the NUT charge and the electromagnetic charges are left invariant, and the mass transforms as follows

$$m(\lambda) = \frac{\lambda + \lambda^{-1}}{2} m + \frac{\lambda - \lambda^{-1}}{2} c \, . \tag{5.70}$$

For $\lambda = \sqrt{\frac{m-c}{m+c}}$ one gets the BPS Reissner–Nordström solution.

6 $\mathcal{N} = 4$ supergravity as an example

Our final example comprises the cases of pure and matter-coupled $\mathcal{N} = 4$ supergravity. We discuss these models in finer detail here mainly in order to illustrate the efficiency of our methods. From table 3 we see that the relevant duality groups are $\mathfrak{G}_4 = \mathrm{SO}(6, n)$, which is enlarged to $\mathfrak{G} = \mathrm{SO}(8, 2+n)$ in the reduction to three dimensions, and where n denotes the number of vector multiplets in four dimensions. In particular, we will analyse the charge matrix $\mathscr{C}$ directly in terms of $\mathrm{Spin}(8, 2)$ for pure $\mathcal{N} = 4$ supergravity.

6.1 The non-linear sigma model formulation

Since we will only consider stationary axisymmetric solutions, it is convenient to use the so-called Weyl coordinates

$$ds^2 = H^{-1} e^{2\sigma} \delta_{\alpha\beta} dx^\alpha dx^\beta + \rho^2 H^{-1} d\varphi^2 - H(dt + \hat{B} d\varphi)^2 \, . \tag{6.1}$$

The bosonic sector of $\mathcal{N} = 4$ supergravity includes six vector fields $U^a dt + \hat{A}^a d\varphi$ which transform in the vector representation of SO(6). They are coupled to scalar fields lying in the coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ [51]. We will write X for the dilaton and Y for the axion field. The two-dimensional action leading to the equations of motion of stationary axisymmetric fields configurations is given by

$$\begin{aligned}\int dx^2 \Big(&-2\partial^\alpha \sigma \partial_\alpha \rho + \frac{1}{2} \rho H^{-2} \partial^\alpha H \partial_\alpha H - \frac{1}{2} \rho^{-1} H^2 \partial^\alpha \hat{B} \partial_\alpha \hat{B} \\ &- \rho H^{-1} X \partial^\alpha U_a \partial_\alpha U^a + \rho^{-1} H X \big(\partial^\alpha \hat{A}_a + U_a \partial^\alpha \hat{B} \big) \big(\partial_\alpha \hat{A}^a + U^a \partial_\alpha \hat{B} \big) \\ &+ \frac{1}{2} \rho X^{-2} \big(\partial^\alpha X \partial_\alpha X + \partial^\alpha Y \partial_\alpha Y \big) + 2\varepsilon^{ij} Y \partial_\alpha U_a \big(\partial_\beta \hat{A}^a + U^a \partial_\beta \hat{B} \big) \Big) \, . \end{aligned} \tag{6.2}$$

This action is invariant with respect to a non-linear representation of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(4) \cong \mathrm{Spin}(2,2) \times \mathrm{Spin}(6)$, where SU(4) is linearly represented on the vector fields as the vector representation of SO(6), and the $\mathrm{SL}(2,\mathbb{R})$'s correspond to an $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1) \times \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ non-linear sigma model. After dualising the fields $\hat{A}^a$ and $\hat{B}$ through the definitions

$$\begin{aligned} \rho^{-1} H X \big(\partial_\alpha \hat{A}^a + U^a \partial_\alpha \hat{B} \big) &= \varepsilon_{\alpha\beta} \big(\partial^\beta A^a + Y \partial^\beta U^a \big) \\ \rho^{-1} H^2 \partial_\alpha \hat{B} &= \varepsilon_{\alpha\beta} \big(\partial^\beta B + U_a \partial^\beta A^a - A_a \partial^\beta U^a \big) \, , \end{aligned} \tag{6.3}$$

the equations of motion of the dual fields follow from the action

$$\int dx^2 \Big(-2\partial^\alpha \sigma \partial_\alpha \rho + \frac{1}{2}\rho \Big(X^{-2}\partial^\alpha X \partial_\alpha X + X^{-2}\partial^\alpha Y \partial_\alpha Y + H^{-2}\partial^\alpha H \partial_\alpha H$$
$$+ H^{-2}\big(\partial^\alpha B + U_a \partial^\alpha A^a - A_a \partial^\alpha U^a\big)\big(\partial_\alpha B + U_a \partial_\alpha A^a - A_a \partial_\alpha U^a\big)$$
$$- 2H^{-1} X \partial^\alpha U_a \partial_\alpha U^a - 2H^{-1}X^{-1}\big(\partial^\alpha A_a + Y\partial^\alpha U_a\big)\big(\partial_\alpha A^a + Y\partial_\alpha U^a\big)\Big)\Big) . \quad (6.4)$$

This action is itself invariant with respect to non-linear transformations of $\mathrm{Spin}(2,8)$, and can be identified as a non-linear sigma model over the coset $\mathrm{SO}(2,8)/\mathrm{SO}(2,6) \times \mathrm{SO}(2)$.

In order to make explicit the four-dimensional character of the solutions in which we are interested, we use a representation of $\mathrm{Spin}(2,8)$ that makes the four-dimensional duality group $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SU}(4)$ explicit, as well as the $\mathrm{SL}(2,\mathbb{R})$ duality group of pure gravity in three dimensions. We thus choose a representation for which the subgroup $\mathrm{Spin}(2,2) \times \mathrm{Spin}(6)$ is block diagonal. This representation is given by matrices valued in the Clifford algebra associated to $\mathbb{R}^6$, which is defined as follows

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab} \qquad \gamma^{ab} \equiv \frac{1}{4}[\gamma^a, \gamma^b]$$
$$C^2 = 1 \qquad C\gamma^a C = -\gamma^{a t} \qquad C\gamma^{ab} C = -\gamma^{ab^t} . \quad (6.5)$$

We thus define the generators of $\mathfrak{spin}(2,8)$ in terms of the six numbers h, e, f, h', e' and f', as well as the four six-dimensional vectors contracted with γ^a, $\not{q}_1$, $\not{p}_1$, $\not{q}_2$, $\not{p}_2$, and the generators of $\mathfrak{spin}(6)$, $\not{v}$. We use the familiar 'slash' notation in order to make clear which objects are Clifford-algebra valued, i.e. $\not{q}_1 \equiv q_{1\,a}\gamma^a$, $\not{v} \equiv \frac{1}{2} v_{ab}\gamma^{ab}$, etc. An element $\mathfrak{u}$ of $\mathfrak{spin}(2,8)$ is parametrised by these submatrices as follows

$$\mathfrak{u} = \begin{pmatrix} h+\not{v} & e & \not{q}_1 & \not{p}_1 \\ f & -h+\not{v} & -\not{p}_2 & \not{q}_2 \\ \not{q}_2 & -\not{p}_1 & h'+\not{v} & e' \\ \not{p}_2 & \not{q}_1 & f' & -h'+\not{v} \end{pmatrix} \quad (6.6)$$

where objects without a slash are to be multiplied by the unit matrix; thus, $\mathfrak{u}$ can be viewed as a complex 16-by-16 or as a real 32-by-32 matrix. We will generally identify the elements of the Clifford algebra proportional to the unit matrix with the real numbers. The subalgebra $\mathfrak{spin}(2,6)\oplus\mathfrak{so}(2)$ is defined by the elements $\boldsymbol{\alpha}$ of $\mathfrak{spin}(2,8)$ satisfying $C\boldsymbol{\alpha}^t C = -\boldsymbol{\alpha}$ (where C is considered as the diagonal four by four matrix with all diagonal entries equal to the Clifford element C), and can be written

$$\boldsymbol{\alpha} = \begin{pmatrix} \not{v} & b & \not{q} & \not{p} \\ -b & \not{v} & -\not{p} & \not{q} \\ \not{q} & -\not{p} & \not{v} & -b \\ \not{p} & \not{q} & b & \not{v} \end{pmatrix} + \begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \end{pmatrix} . \quad (6.7)$$

We define the coset representative $\mathcal{V}$ with generators $\mathbf{h}$ and $\mathbf{e}$ for the gravity fields H and B, $\mathbf{h}'$ and $\mathbf{e}'$ for the dilaton X and the axion Y, and $\mathbf{q}^a$ for the six electric fields U^a and

$\mathbf{p}^a$ for the six magnetic fields A^a. It is given by the matrix

$$\mathcal{V} = \begin{pmatrix} H^{\frac{1}{2}} & H^{-\frac{1}{2}}\bigl(B - \frac{1}{2}[\not{\Psi}, \not{A}]\bigr) & X^{\frac{1}{2}}\not{\Psi} & X^{-\frac{1}{2}}\bigl(\not{A} + Y\not{\Psi}\bigr) \\ 0 & H^{-\frac{1}{2}} & 0 & 0 \\ 0 & -H^{-\frac{1}{2}}\not{A} & X^{\frac{1}{2}} & X^{-\frac{1}{2}}Y \\ 0 & H^{-\frac{1}{2}}\not{\Psi} & 0 & X^{-\frac{1}{2}} \end{pmatrix} . \tag{6.8}$$

The component of $\mathcal{V}^{-1}d\mathcal{V}$ lying in the orthogonal complement of $\mathfrak{spin}(2,6) \oplus \mathfrak{so}(2)$ inside $\mathfrak{spin}(2,8)$ is given by

$$2P \equiv \mathcal{V}^{-1}d\mathcal{V} + C(\mathcal{V}^{-1}d\mathcal{V})^t C =$$
$$\begin{pmatrix} H^{-1}dH & H^{-1}\bigl(dB + \frac{1}{2}\{\not{\Psi}, d\not{A}\} - \frac{1}{2}\{\not{A}, d\not{\Psi}\}\bigr) & \bigl(\frac{X}{H}\bigr)^{\frac{1}{2}} d\not{\Psi} & (HX)^{-\frac{1}{2}}\bigl(d\not{A} + Y d\not{\Psi}\bigr) \\ H^{-1}\bigl(dB + \frac{1}{2}\{\not{\Psi}, d\not{A}\} - \frac{1}{2}\{\not{A}, d\not{\Psi}\}\bigr) & -H^{-1}dH & (HX)^{-\frac{1}{2}}\bigl(d\not{A} + Y d\not{\Psi}\bigr) & -\bigl(\frac{X}{H}\bigr)^{\frac{1}{2}} d\not{\Psi} \\ -\bigl(\frac{X}{H}\bigr)^{\frac{1}{2}} d\not{\Psi} & -(HX)^{-\frac{1}{2}}\bigl(d\not{A} + Y d\not{\Psi}\bigr) & X^{-1}dX & X^{-1}dY \\ -(HX)^{-\frac{1}{2}}\bigl(d\not{A} + Y d\not{\Psi}\bigr) & \bigl(\frac{X}{H}\bigr)^{\frac{1}{2}} d\not{\Psi} & X^{-1}dY & -X^{-1}dX \end{pmatrix} \tag{6.9}$$

in such a way that the action (6.4) is given by

$$\int dx^2 \Bigl(-2\partial^\alpha \sigma \partial_\alpha \rho + \rho \mathrm{Tr}\, P_\alpha P^\alpha \Bigr) . \tag{6.10}$$

6.2 $\mathrm{SO}(2,6) \times \mathrm{SO}(2)$-orbits of solutions

The simplest Reissner–Nordström like solutions of the $\mathcal{N} = 4$ theory are the ones for which the axion field is identically zero [53]. The vector source term for the axion field then obeys

$$\{\partial^\alpha \not{\Psi}, \partial_\alpha \not{A}\} - \{\partial^\alpha \not{A}, \partial_\alpha \not{\Psi}\} = 0 . \tag{6.11}$$

These solutions have electric and magnetic charge vectors which are orthogonal in $\mathbb{R}^6$. They can be obtained from the Schwarzschild solution by the following $\mathrm{Spin}(2,6)$ transformation

$$u(\not{p}, \not{q}) = \frac{1}{\sqrt{(1-p^2)(1-q^2)}} \begin{pmatrix} 1 & -\not{q}\not{p} & \not{q} & \not{p} \\ \not{q}\not{p} & 1 & -\not{p} & \not{q} \\ \not{q} & -\not{p} & 1 & \not{q}\not{p} \\ \not{p} & \not{q} & -\not{q}\not{p} & 1 \end{pmatrix} \tag{6.12}$$

with $\{\not{q}, \not{p}\} = 0$. The action of $u(\not{p}, \not{q})$ on the Schwarzschild matrix v_0 of mass $m = c$ gives the dilaton black hole matrix $\mathcal{V}$ through

$$\mathcal{V} = u(\not{p}, \not{q})\, v_0\, u\left(-\sqrt{\tfrac{r-c}{r+c}}\, \not{p}, -\sqrt{\tfrac{r-c}{r+c}}\, \not{q} \right) . \tag{6.13}$$

The dilaton black hole then has mass $M = \frac{1-q^2p^2}{(1-q^2)(1-p^2)}c$, electric charge $Q^a = \frac{q^a}{1-q^2}c$ and magnetic charge $P^a = \frac{p^a}{1-p^2}c$ with $P_aQ^a = 0$, and dilaton charge $\Sigma = \frac{\slashed{P}^2-\slashed{Q}^2}{M}$. The BPS parameter is given by the formula

$$c^2 = M^2 - 2\slashed{Q}^2 - 2\,\slashed{P}^2 + \Sigma^2\,, \tag{6.14}$$

while the coset representative $\mathcal{V}$ is

$$\mathcal{V} = \begin{pmatrix} \sqrt{\frac{r^2-c^2}{(r+M)^2-\Sigma^2}} & \frac{-2[\slashed{Q}\,,\slashed{P}\,]}{\sqrt{r^2-c^2}\sqrt{(r+M)^2-\Sigma^2}} & \frac{2\slashed{Q}}{\sqrt{(r+M)^2-\Sigma^2}} & \frac{2\,\slashed{P}}{\sqrt{(r+M)^2-\Sigma^2}} \\ 0 & \sqrt{\frac{(r+M)^2-\Sigma^2}{r^2-c^2}} & 0 & 0 \\ 0 & -\frac{2\,\slashed{P}}{\sqrt{r^2-c^2}}\sqrt{\frac{r+M-\Sigma}{r+M+\Sigma}} & \sqrt{\frac{r+M-\Sigma}{r+M+\Sigma}} & 0 \\ 0 & \frac{2\slashed{Q}}{\sqrt{r^2-c^2}}\sqrt{\frac{r+M+\Sigma}{r+M-\Sigma}} & 0 & \sqrt{\frac{r+M+\Sigma}{r+M-\Sigma}} \end{pmatrix}. \tag{6.15}$$

The non-linear SO(2) action of the $SL(2,\mathbb{R})$ dilaton-axion sigma model permits one to obtain the general solution for arbitrary electric and magnetic charges and with a non-trivial axion field. The non-linear SO(2) of the pure gravity $SL(2,\mathbb{R})$ sigma model turns on the NUT charge, just as for pure gravity. For general solutions [52], the dilaton and axion charges Σ and Ξ are given by[28]

$$\Sigma = \frac{(\slashed{P}^2 - \slashed{Q}^2)M + \{\slashed{Q},\slashed{P}\}N}{M^2+N^2} \qquad \Xi = \frac{\{\slashed{Q},\slashed{P}\}M - (\slashed{P}^2 - \slashed{Q}^2)N}{M^2+N^2}\,. \tag{6.16}$$

The $\mathfrak{so}(2,8) \ominus (\mathfrak{so}(2) \oplus \mathfrak{so}(2,6))$ charge matrix $\mathscr{C}$ is then given by

$$\mathscr{C} = \begin{pmatrix} M & -N & -\slashed{Q} & -\slashed{P} \\ -N & -M & -\slashed{P} & \slashed{Q} \\ \slashed{Q} & \slashed{P} & \Sigma & -\Xi \\ \slashed{P} & -\slashed{Q} & -\Xi & -\Sigma \end{pmatrix} \tag{6.17}$$

(thus justifying our definition of the electric and magnetic charges including a factor $\sqrt{2}$ with respect to the usual one [53]). The BPS parameter in the SO(2,6) basis is given by

$$c^2 \equiv \frac{1}{16}\mathrm{Tr}\,\mathscr{C}^2 = M^2 + N^2 - 2\slashed{Q}^2 - 2\slashed{P}^2 + \Sigma^2 + \Xi^2\,. \tag{6.18}$$

Using the explicit form of the charge matrix, the cubic equation $\mathscr{C}^3 = c^2\mathscr{C}$ is perfectly equivalent to the complex equation

$$(M+iN)(\Sigma+i\Xi) = (\slashed{P} + i\slashed{Q})^2\,, \tag{6.19}$$

from which the expression (6.16) for the scalar charges can be derived. To establish a link with the notation of the preceding sections, one must use the isomorphism $\mathrm{Spin}(6) \cong SU(4)$

[28] Recall that we identify the elements of $Cl(6,\mathbb{R})$ proportional to the unit matrix $\mathbb{1}$ with ordinary real numbers.

and the fact that the matrices $i[C\gamma^a]_{ij}$ define a basis for the complex self-dual antisymmetric tensors of SU(4). We have that $Z_{ij} \equiv [C(\not{P} + i\not{Q})]_{ij}$ and $\Sigma_{ijkl} \equiv \frac{1}{4!}\varepsilon_{ijkl}(\Sigma + i\Xi)$. As explained in section 3.3, equation (6.19) is in fact the Spin*(8) pure spinor equation, which corresponds within $\mathrm{U}(1) \times \mathrm{SO}(6,2)$ to the fact that the charge matrix defines a complex null vector.

Because (6.19) is invariant under the action of $\mathrm{Spin}(2,6) \times \mathrm{SO}(2)$, its solutions define non-linear representations of $\mathrm{SO}(2,6) \times \mathrm{SO}(2)$. The maximal compact subgroup $\mathrm{U}(4) \times \mathrm{SO}(2)$ is linearly realised on M, N, $\not{Q}$ and $\not{P}$. U(4) acts only on $\not{Q} + i\not{P}$ as it does on the vector fields, and SO(2) rotates $\not{P}$ into $\not{Q}$ and M into N with doubled weight for the latter. The non-compact elements act non-linearly in the following way

$$\begin{aligned}
M(q) &= \frac{1}{1-q^2}M - \frac{q^2}{1-q^2}\Sigma + \frac{1}{1-q^2}\{\not{q}, \not{Q}\} & \not{Q}(q) &= \frac{\not{Q} + \not{q}\not{Q}\not{q}}{1-q^2} + \frac{\not{q}}{1-q^2}(M - \Sigma) \\
N(q) &= \frac{1}{1-q^2}N - \frac{q^2}{1-q^2}\Xi - \frac{1}{1-q^2}\{\not{q}, \not{P}\} & \not{P}(q) &= \frac{\not{P} + \not{q}\not{P}\not{q}}{1-q^2} - \frac{\not{q}}{1-q^2}(N - \Xi) \\
M(p) &= \frac{1}{1-p^2}M + \frac{p^2}{1-p^2}\Sigma + \frac{1}{1-p^2}\{\not{p}, \not{P}\} & \not{Q}(p) &= \frac{\not{Q} + \not{p}\not{Q}\not{p}}{1-p^2} + \frac{\not{p}}{1-p^2}(N + \Xi) \\
N(p) &= \frac{1}{1-p^2}N + \frac{p^2}{1-p^2}\Xi + \frac{1}{1-p^2}\{\not{p}, \not{Q}\} & \not{P}(p) &= \frac{\not{P} + \not{p}\not{P}\not{p}}{1-p^2} + \frac{\not{p}}{1-p^2}(M + \Sigma) \, .
\end{aligned} \tag{6.20}$$

For a non-zero fixed value of the BPS parameter c, this gives an irreducible representation of $\mathrm{SO}(2,6) \times \mathrm{SO}(2)$ on which this group acts transitively. One can see explicitly that the moduli spaces of spherically symmetric $\frac{1}{4}$ BPS and $\frac{1}{2}$ BPS Taub–NUT black holes (i.e. $\mathcal{M}_1$ and $\mathcal{M}_2$) define distinct $\mathrm{SO}(2,6) \times \mathrm{SO}(2)$-orbits from the factorisation of the BPS parameter square c^2 into

$$c^2 = \left(\sqrt{M^2 + N^2} - \frac{\not{Q}^2 + \not{P}^2 + \sqrt{-[\not{Q}, \not{P}]^2}}{\sqrt{M^2 + N^2}} \right) \left(\sqrt{M^2 + N^2} - \frac{\not{Q}^2 + \not{P}^2 - \sqrt{-[\not{Q}, \not{P}]^2}}{\sqrt{M^2 + N^2}} \right) . \tag{6.21}$$

If only one of these factors is zero, the solution becomes $\frac{1}{4}$ BPS, and if both of them are zero (without the solution being trivial), it becomes one-half BPS. For the $\frac{1}{4}$ BPS case, we consider in fact only the situation where the smaller factor is vanishing, so as to respect the positivity of the Bogomolny bounds. The compact subgroup $\mathrm{U}(4) \times \mathrm{SO}(2)$ leaves invariant each of these factors. Since the linear $\mathrm{SO}(2) \times \mathrm{SO}(2)$ acts freely on $\Sigma + i\Xi$ and $M + iN$, one can restrict oneself to the action of the non-compact generators for a dilaton black hole with $N = \Xi = 0$. In this case, one can write $\not{P}$ and $\not{Q}$ as numbers, and the non-compact generators then act non trivially only if q^a is in the direction of Q^a and respectively if p^a is in the direction of P^a. In this case, the Lie algebra action for the generators $\mathbf{p}$ and $\mathbf{q}$ on the two factors is

$$\begin{aligned}
\mathbf{q} \quad &: \qquad \delta\left(M - \frac{(Q \pm P)^2}{M} \right) = \mp \frac{2P}{M} \left(M - \frac{(Q \pm P)^2}{M} \right) \\
\mathbf{p} \quad &: \qquad \delta\left(M - \frac{(Q \pm P)^2}{M} \right) = \mp \frac{2Q}{M} \left(M - \frac{(Q \pm P)^2}{M} \right) \, .
\end{aligned} \tag{6.22}$$

We see that the action of $\mathrm{SO}(2,6)\times\mathrm{SO}(2)$ on the two factors is a non-linear rescaling. Thus, this action leaves invariant the number of preserved supersymmetry charges of a given solution. We conclude that the irreducible representation of $\mathrm{SO}(2,6)\times\mathrm{SO}(2)$ for a non-zero value of c decomposes for vanishing c into three irreducible representations, which are the $\frac{1}{4}$ BPS set, the $\frac{1}{2}$ BPS set and the fully BPS Minkowski singlet, as stated in section 4.1.

Let us now describe the coset decomposition of the space of solutions. The product group of the trombone symmetry $\mathbb{R}_+^*$ and $\mathrm{SO}(2,6)\times\mathrm{SO}(2)$ acts transitively on non-BPS solutions for a fixed value of $\frac{a}{c}$. Since the subgroup leaving a pure gravity solution invariant is the four-dimensional duality group, such an orbit takes the form $\big(\mathbb{R}_+^*\times\mathrm{SO}(2,6)\times\mathrm{SO}(2)\big)/\mathrm{SO}(2)\times\mathrm{SO}(6)$. There are actually non-BPS solutions with a positive value of c^2 that do not lie on the Schwarzschild orbit. These can be obtained from the orbit of a purely dilatonic solution for which all the charges are zero except for the dilaton charge Σ. Such a charge obviously satisfies (6.19). The metric is then given by

$$ds^2=\frac{r^2-\Sigma^2}{r_+r_-}\big(dz^2+d\rho^2\big)+\rho^2d\varphi^2-dt^2 \tag{6.23}$$

and the associated Ricci scalar is $\mathcal{R}=\frac{2\Sigma^2}{(r^2-\Sigma^2)^2}$. Note that this solution has a naked singularity. In fact, all the solutions of the corresponding $\mathrm{SO}(2,6)\times\mathrm{SO}(2)$-orbit violate simultaneously the two Bogomolny bounds and so this orbit consists entirely of badly behaved solutions and will be disregarded.

For BPS solutions, the action of the trombone is identified with the action of one of the generators of $\mathrm{SO}(2,6)\times\mathrm{SO}(2)$. It is enough to compute the isotropy subgroup for a particular solution. Starting from a $\frac{1}{4}$ BPS solution with $\{\not{Q},\not{P}\}=0$ and $N=0$, the $\mathfrak{spin}(2,6)\oplus\mathfrak{so}(2)$ elements commuting with the charge matrix take the following form[29]

$$\begin{pmatrix} \not{\psi} & a+\frac{\{\not{Q},[\not{\psi},\not{P}]\}-4(\not{Q}^2+\not{P}^2)a}{4QP} & -\frac{2a\not{P}+[\not{\psi},\not{Q}]}{2Q} & \frac{2a\not{Q}-[\not{\psi},\not{P}]}{2P} \\ -a-\frac{\{\not{Q},[\not{\psi},\not{P}]\}-4(\not{Q}^2+\not{P}^2)a}{4QP} & \not{\psi} & -\frac{2a\not{Q}-[\not{\psi},\not{P}]}{2P} & -\frac{2a\not{P}+[\not{\psi},\not{Q}]}{2Q} \\ -\frac{2a\not{P}+[\not{\psi},\not{Q}]}{2Q} & -\frac{2a\not{Q}-[\not{\psi},\not{P}]}{2P} & \not{\psi} & a-\frac{\{\not{Q},[\not{\psi},\not{P}]\}-4(\not{Q}^2+\not{P}^2)a}{4QP} \\ \frac{2a\not{Q}-[\not{\psi},\not{P}]}{2P} & -\frac{2a\not{P}+[\not{\psi},\not{Q}]}{2Q} & -a+\frac{\{\not{Q},[\not{\psi},\not{P}]\}-4(\not{Q}^2+\not{P}^2)a}{4QP} & \not{\psi} \end{pmatrix}. \tag{6.24}$$

We define the indices i, j, $\cdots$ as the $\mathrm{SO}(6)$ indices orthogonal to both $\not{Q}$ and $\not{P}$, and we take 1 and 2 as the index values for these directions. With the redefinitions

$$\upsilon^\alpha{}_\beta\equiv\varepsilon^\alpha{}_\beta a \qquad z\equiv\upsilon_{12} \qquad x_i^\alpha\equiv(\upsilon_{i1},\upsilon_{i2}) \tag{6.25}$$

where $\varepsilon_{\alpha\beta}$ is the $\mathrm{SO}(2)$ antisymmetric invariant tensor, the corresponding generators have

[29] Note that $\not{Q}$ and $\not{P}$ are both necessarily non-zero for a strictly $\frac{1}{4}$ BPS solution.

the following non-vanishing commutators

$$
\begin{gathered}
[\boldsymbol{v}_i{}^j, \boldsymbol{v}_k{}^l] = 2\delta^{[j}_{[k} \boldsymbol{v}_{i]}{}^{l]} \\
[\boldsymbol{v}^\alpha{}_\beta, \boldsymbol{x}^\gamma_i] = \delta^\gamma_\beta \boldsymbol{x}^\alpha_i \qquad\qquad [\boldsymbol{v}_i{}^j, \boldsymbol{x}^\alpha_k] = \delta^j_k \boldsymbol{x}^\alpha_i \\
[\boldsymbol{x}^\alpha_i, \boldsymbol{x}^\beta_j] = \delta_{ij} \varepsilon^{\alpha\beta} \boldsymbol{z} \ .
\end{gathered} \tag{6.26}
$$

We will call this algebra $\mathfrak{ic}(\mathfrak{so}(2) \oplus \mathfrak{so}(4))$, i.e. this is the Poincaré-like algebra $\mathfrak{i}(\mathfrak{so}(2) \oplus \mathfrak{so}(4))$ with a central charge, and with corresponding group $Ic(\mathrm{SO}(2) \times \mathrm{SO}(4))$.

Purely electric dilatonic $\frac{1}{2}$ BPS black holes have a charge matrix of the form

$$
\mathscr{C} = \begin{pmatrix} Q & 0 & -\not{Q} & 0 \\ 0 & -Q & 0 & \not{Q} \\ \not{Q} & 0 & -Q & 0 \\ 0 & -\not{Q} & 0 & Q \end{pmatrix} \ . \tag{6.27}
$$

One can easily check that this matrix satisfies $\mathscr{C}^2 = 0$. The $\mathfrak{spin}(2,6) \oplus \mathfrak{so}(2)$ elements that commute with this charge matrix are of the following form

$$
\begin{pmatrix}
\not{v} & -\frac{\{\not{p}, \not{Q}\}}{2Q} & -\frac{[\not{v}, \not{Q}]}{2Q} & \not{p} \\
\frac{\{\not{p}, \not{Q}\}}{2Q} & \not{v} & -\not{p} & -\frac{[\not{v}, \not{Q}]}{2Q} \\
-\frac{[\not{v}, \not{Q}]}{2Q} & -\not{p} & \not{v} & \frac{\{\not{p}, \not{Q}\}}{2Q} \\
\not{p} & -\frac{[\not{v}, \not{Q}]}{2Q} & -\frac{\{\not{p}, \not{Q}\}}{2Q} & \not{v}
\end{pmatrix} \ . \tag{6.28}
$$

These elements generate the six-dimensional Poincaré algebra $\mathfrak{iso}(1,5)$, where the components of $\boldsymbol{v}$ and $\boldsymbol{p}$ orthogonal to $\not{Q}$ generate $\mathfrak{so}(1,5)$ and their components collinear to $\not{Q}$ generate the abelian subalgebra $\mathbb{R}^6$.

Finally, the space of asymptotically flat particle-like stationary solutions has the following decomposition into $\mathrm{SO}(2,6) \times \mathrm{SO}(2)$-orbits

$$
[-1,1] \times \frac{\mathbb{R}^*_+ \times \mathrm{SO}(2,6) \times \mathrm{SO}(2)}{\mathrm{SO}(2) \times \mathrm{SO}(6)} \cup \frac{\mathrm{SO}(2,6) \times \mathrm{SO}(2)}{Ic(\mathrm{SO}(2) \times \mathrm{SO}(4))} \cup \frac{\mathrm{SO}(2,6) \times \mathrm{SO}(2)}{I\mathrm{SO}(1,5)} \cup \{0\} \tag{6.29}
$$

where $[-1,1]$ stands for the angular momentum per unit of mass, in perfect agreement with the results of section 4.1.

6.3 $\mathcal{N} = 4$ supergravity coupled to n vector multiplets

Let us consider briefly the more general case of $\mathcal{N} = 4$ supergravity coupled to n vector multiplets. We will just give here the main results without explaining the full details. The scalar fields of the corresponding non-linear sigma model lie in the coset space $\mathrm{Spin}(8, 2+n)/(\mathrm{SO}(6,2) \times \mathrm{SO}(2,n))$ and the charge matrix $\mathscr{C}$ can be represented as a Majorana–Weyl chiral spinor of $\mathrm{Spin}^*(8) \cong \mathrm{Spin}(6,2)$ valued in the vector representation of $\mathrm{SO}(2,n)$

$$
|\mathscr{C}\rangle \equiv \begin{pmatrix} \bigl(W + Z_{ij}\ a^i a^j + \frac{1}{4}\varepsilon_{ijkl}\, \Sigma\ a^i a^j a^k a^l\bigr)|0\rangle \\ \bigl(z^A + \Sigma^A_{ij+} a^i a^j + \frac{1}{4}\varepsilon_{ijkl}\, \bar{z}^A\, a^i a^j a^k a^l\bigr)|0, A\rangle \end{pmatrix} , \tag{6.30}
$$

where the index A lies in the vector representation of $\mathrm{SO}(n)$. Note that only the $\mathrm{SO}(n)$ vector components obey the $\mathrm{Spin}^*(8)$ self-duality constraint, while the first two components of the $\mathrm{SO}(2,n)$ vector have been combined into a complex state. The 'Dirac equation' (2.46) gives the same constraints on W, Z_{ij} and Σ as in the pure supergravity case and furthermore we have it that

$$\Big(\Sigma^A_{ij_+} - \frac{z^A Z_{ij}}{W}\Big)\epsilon^j_\alpha = 0 \tag{6.31}$$

from which one can derive the $\frac{1}{4}$ and the $\frac{1}{2}$ BPS conditions. These solutions have been derived in [54, 55].

It follows from the 3-graded decomposition of the spinor representation of $\mathfrak{spin}(8,2+n)$ that the cubic constraint, $\mathscr{C}^3 = c^2\mathscr{C}$, must be satisfied in the spinor representation, which implies its validity in the vector representation. Its components bilinear in the gamma matrices of $\mathfrak{spin}(6,2)$ and $\mathfrak{spin}(2,n)$ yield a component of $\mathscr{C}\otimes\mathscr{C}$ in the symmetric traceless rank two tensor representation of $\mathrm{SO}(2,n)$ which vanishes, and its component bi-linear in the antisymmetric product of three gamma matrices of $\mathfrak{spin}(6,2)$ and $\mathfrak{spin}(2,n)$ yields a component of $\mathscr{C}\otimes\mathscr{C}\otimes\mathscr{C}$ in the product of the antisymmetric rank three tensor representation of $\mathrm{SO}(6,2)$ times the antisymmetric rank three tensor representation of $\mathrm{SO}(2,n)$ which vanishes too, i.e.

$$\eta_{\mathcal{I}\mathcal{J}}\,\mathscr{C}^{\mathcal{I}}_{\mathcal{A}}\mathscr{C}^{\mathcal{J}}_{\mathcal{B}} = \frac{1}{8}\eta_{\mathcal{A}\mathcal{B}}\,\eta^{\mathcal{C}\mathcal{D}}\,\eta_{\mathcal{I}\mathcal{J}}\,\mathscr{C}^{\mathcal{I}}_{\mathcal{C}}\mathscr{C}^{\mathcal{J}}_{\mathcal{D}} \qquad\qquad \mathscr{C}^{[\mathcal{I}}_{[\mathcal{A}}\mathscr{C}^{\mathcal{J}}_{\mathcal{B}}\mathscr{C}^{\mathcal{K}]}_{\mathcal{C}]} = 0\ . \tag{6.32}$$

where $\mathcal{I}$, $\mathcal{J}$, $\cdots$ and $\mathcal{A}$, $\mathcal{B}$, $\cdots$ lie in the vector representation of $\mathrm{SO}(6,2)$ and $\mathrm{SO}(2,n)$, respectively, and $\eta_{\mathcal{I}\mathcal{J}}$ and $\eta_{\mathcal{A}\mathcal{B}}$ are the corresponding invariant tensors. The general solution is a non-rational function of W, Z_{ij} and z^A, but one can nevertheless determine the general solution by using the transitivity of $\mathrm{SO}(6,2)\times\mathrm{SO}(2,n)$ on non-extremal solutions. A general non-extremal solution ($c^2>0$) can indeed be obtained by acting with a general $\mathrm{SO}(2,n)$ element on a general $\mathrm{Spin}^*(8)$ pure spinor

$$|\mathscr{C}\rangle \equiv \begin{pmatrix} \big(\mathrm{x} + X_{ij}a^ia^j + \frac{1}{2\mathrm{x}}X_{ij}X_{kl}a^ia^ja^ka^l\big)|0\rangle \\ 0 \end{pmatrix}\ . \tag{6.33}$$

The $\mathrm{SO}(2,n)$ element can be chosen to be the product of an $\mathrm{SO}(2)$ rotation and transformations generated by two orthogonal non compact generators. It gives the general non-extremal solution of (6.32) as

$$\begin{aligned}
W &= e^{i\alpha}\Big(\frac{\cosh u + \cosh v}{2}\mathrm{x} + \frac{\cosh u - \cosh v}{2}\frac{1}{2\bar{\mathrm{x}}}\varepsilon_{ijkl}X^{ij}X^{kl}\Big)\\
Z_{ij} &= e^{i\alpha}\Big(\frac{\cosh u + \cosh v}{2}X_{ij} + \frac{\cosh u - \cosh v}{2}\frac{1}{2}\varepsilon_{ijkl}X^{kl}\Big)\\
\Sigma &= e^{i\alpha}\Big(\frac{\cosh u + \cosh v}{2}\frac{1}{2\mathrm{x}}\varepsilon^{ijkl}X_{ij}X_{kl} + \frac{\cosh u - \cosh v}{2}\bar{\mathrm{x}}\Big)\\
z^A &= \frac{1}{2}\hat{u}^A\sinh u\Big(\mathrm{x} + \frac{1}{2\bar{\mathrm{x}}}\varepsilon_{ijkl}X^{ij}X^{kl}\Big) + \frac{i}{2}\hat{v}^A\sinh v\Big(\mathrm{x} - \frac{1}{2\bar{\mathrm{x}}}\varepsilon_{ijkl}X^{ij}X^{kl}\Big)\\
\Sigma^A_{ij_+} &= \hat{u}^A\sinh u X_{ij_+} + i\hat{v}^A\sinh v X_{ij_-}\ ,
\end{aligned} \tag{6.34}$$

where $\hat{u}^A$ and $\hat{v}^A$ are real orthogonal SO(n) vectors of norm one. Non-BPS extremal solutions correspond to the limit where the SO$(2,n)$ element goes to the SO$(2,n)$ boundary, that is when either u, or v, or both go to infinity. The generic case corresponds to the limit where both go to infinity in such a way that $e^u - e^v$ remains finite. The corresponding non-BPS extremal solutions satisfy

$$\Sigma = \frac{\bar{z}^A \bar{z}_A}{\bar{W}} = \frac{1}{2W} \varepsilon^{ijkl} Z_{ij} Z_{kl} \qquad\qquad \bar{z}_A z^A = |W|^2 + |\Sigma|^2 \, . \tag{6.35}$$

Finally, there are two distinguished cases, either where $|z^A z_A| < |W|^2$, in which case the solution remains rather complicated in general, or where $|z^A z_A| = |W|^2$, in which case

$$\Sigma^A_{ij+} = \frac{z^A Z_{ij}}{W} = \frac{1}{2} \varepsilon_{ijkl} \frac{\bar{z}^A Z^{kl}}{\bar{W}} \, . \tag{6.36}$$

The strata are[30]

$$\begin{aligned}
\mathcal{M}_{(0,0)} &\cong \mathbb{R}^*_+ \times \frac{\mathrm{SO}(6,2)\times \mathrm{SO}(2,n)}{\mathrm{SO}(2)\times \mathrm{SO}(6)\times \mathrm{SO}(n)} \\
\mathcal{M}_{(1,0)} &\cong \frac{\mathrm{SO}(6,2)\times \mathrm{SO}(2,n)}{I\!c(\mathrm{SO}(4)\times \mathrm{SO}(2))\times \mathrm{SO}(n)} \qquad \mathcal{M}_{(0,1)} \cong \frac{\mathrm{SO}(6,2)\times \mathrm{SO}(2,n)}{\mathrm{SO}(6)\times I\!c(\mathrm{SO}(n-2)\times \mathrm{SO}(2))} \\
\mathcal{M}_{(1,0)^\circ} &\cong \frac{\mathrm{SO}(6,2)\times \mathrm{SO}(2,n)}{\big(\mathbb{R}^*_+\times \mathrm{SO}(4)\times \mathrm{SO}(n-1)\big)\ltimes\big((\mathbf{1}\oplus\mathbf{4}\oplus\mathbf{n-1})^{(1)}\oplus\mathbf{4}^{(2)}\oplus\mathbf{1}^{(3)}\big)} \\
\mathcal{M}_{(0,1)^\circ} &\cong \frac{\mathrm{SO}(6,2)\times \mathrm{SO}(2,n)}{\big(\mathbb{R}^*_+\times \mathrm{SO}(5)\times \mathrm{SO}(n-2)\big)\ltimes\big((\mathbf{1}\oplus\mathbf{5}\oplus\mathbf{n-2})^{(1)}\oplus\mathbf{n-2}^{(2)}\oplus\mathbf{1}^{(3)}\big)} \\
\mathcal{M}_{(1,1)} &\cong \frac{\mathrm{SO}(6,2)\times \mathrm{SO}(2,n)}{(GL(2,\mathbb{R})\times \mathrm{SO}(4)\times \mathrm{SO}(n-2))\ltimes\big(\mathbf{1}^{(-2)}\oplus\bar{\mathbf{2}}^{(-1)}\otimes\mathbf{4}\oplus\mathbf{2}^{(1)}\otimes\mathbf{n-2}\oplus\mathbf{1}^{(2)}\big)} \\
\mathcal{M}_{(2,0)} &\cong \frac{\mathrm{SO}(6,2)\times \mathrm{SO}(2,n)}{I\mathrm{SO}(5,1)\times \mathrm{SO}(1,n)} \mathcal{M}_{(0,2)} \cong \frac{\mathrm{SO}(6,2)\times \mathrm{SO}(2,n)}{\mathrm{SO}(6,1)\times I\mathrm{SO}(1,n-1)} \\
\mathcal{M}_{(2,2)} &\cong \frac{\mathrm{SO}(6,2)\times \mathrm{SO}(2,n)}{(\mathrm{SO}(1,1)\times \mathrm{SO}(5,1)\times \mathrm{SO}(1,n-1))\ltimes(\mathbf{6}^{(-1)}\oplus\mathbf{n}^{(1)})}
\end{aligned} \tag{6.37}$$

where the stratum $\mathcal{M}_{(p,q)}$ corresponds to solutions which are $\frac{p}{4}$ BPS, and $\mathcal{M}_{(p,q)} \subset \overline{\mathcal{M}}_{(r,s)}$ if and only if both $p \geq r$ and $q \geq s$ (with, in addition, $\partial\mathcal{M}_{(1,0)} = \overline{\mathcal{M}}_{(1,0)^\circ}$ and $\partial\mathcal{M}_{(0,1)} = \overline{\mathcal{M}}_{(0,1)^\circ}$). The properties of the strata are summarised in are summarised in table 9 below.

This stratification is in agreement with the stratification of the nilpotent orbits $\mathfrak{N}_{\mathrm{SO}(8,2+n)}$ of SO$(8,2+n)$ as described in [56, 57]. Nevertheless, for $n \geq 2$, the stratification of $\mathfrak{N}_{\mathrm{SO}(8,2+n)}$ suggests that there is an additional stratum of charge matrices which correspond to extremal black holes without any saturated central charge

$$\mathcal{M}_{(0,0)^\circ} \cong \frac{\mathrm{SO}(6,2)\times \mathrm{SO}(2,n)}{I\mathrm{SO}(5)\times I\mathrm{SO}(n-1)\times \mathbb{R}} \tag{6.38}$$

[30] Note, however, that $\mathcal{M}_{(0,1)}$, $\mathcal{M}_{(0,1)^\circ}$ and $\mathcal{M}_{(1,1)}$ are empty in the case $n=1$, and note that $I\mathrm{SO}(1)$ must be understood as the abelian translation group $\mathbb{R}$. $I\!c(\mathrm{SO}(n-2)\times\mathrm{SO}(2))$ is $\mathrm{SO}(2)\times\mathbb{R}$ for $n=2$ and is $I\!c\mathrm{SO}(2)$ for $n=3$. Moreover, $\mathcal{M}_{(0,1)}$, $\mathcal{M}_{(0,1)^\circ}$ and $\mathcal{M}_{(1,1)}$ have two connected components in the case $n=2$, which can be transformed into one another by $O(2,2)$ parity.

	dim	nilpotency	Horizon area
$\mathcal{M}_{(0,0)}$	$14+2n$	$\mathscr{C}^3 = c^2\mathscr{C} \quad [\mathscr{C},[\mathscr{C},[\mathscr{C},\Gamma^{\mathcal{M}}]]] = c^2[\mathscr{C},\Gamma^{\mathcal{M}}]$	$A>0$
$\mathcal{M}_{(1,0)}$, $\mathcal{M}_{(0,1)}$	$13+2n$	$\mathscr{C}^3 = 0 \quad [\mathscr{C},[\mathscr{C},[\mathscr{C},\Gamma^{\mathcal{M}}]]] = 0 \quad \mathrm{ad}_{\mathscr{C}}{}^5 = 0$	$A>0$
$\mathcal{M}_{(1,0)^\circ}$, $\mathcal{M}_{(0,1)^\circ}$	$12+2n$	$\mathscr{C}^3 = 0 \quad [\mathscr{C},[\mathscr{C},[\mathscr{C},\Gamma^{\mathcal{M}}]]] = 0 \quad \mathrm{ad}_{\mathscr{C}}{}^4 = 0$	$A=0$
$\mathcal{M}_{(1,1)}$	$10+2n$	$\mathscr{C}^3 = 0 \quad [\mathscr{C},[\mathscr{C},\Gamma^{\mathcal{M}}]] = 0 \quad \mathrm{ad}_{\mathscr{C}}{}^3 = 0$	$A=0$
$\mathcal{M}_{(2,0)}$, $\mathcal{M}_{(0,2)}$	$8+n$	$\mathscr{C}^2 = 0 \quad [\mathscr{C},[\mathscr{C},[\mathscr{C},\Gamma^{\mathcal{M}}]]] = 0 \quad \mathrm{ad}_{\mathscr{C}}{}^3 = 0$	$A=0$
$\mathcal{M}_{(2,2)}$	$7+n$	$\mathscr{C}^2 = 0 \quad [\mathscr{C},[\mathscr{C},\Gamma^{\mathcal{M}}]] = 0 \quad \mathrm{ad}_{\mathscr{C}}{}^3 = 0$	$A=0$

Table 9. Dimension of strata in $\mathcal{N}=4$ supergravity with n vector multiplets.

which satisfies the ordering $\mathcal{M}_{(1,0)^\circ} \cup \mathcal{M}_{(0,1)^\circ} \subset \overline{\mathcal{M}}_{(0,0)^\circ} \subset \overline{\mathcal{M}}_{(0,0)}$. Such solutions do indeed exist, some examples of which having been found within the *STU* model [34].

The fact that spherically symmetric extremal solutions of $\mathcal{N}=4$ supergravity are associated to nilpotent orbits of $\mathrm{SO}(8,2+n)$ has already been discussed in [58]. Note that although the $\frac{1}{4}$ BPS solutions are naturally related to the complex geometry of twistor spaces, this is not necessarily the case for the $\frac{1}{2}$ BPS ones. For instance, our analysis (although not yet complete) leads us to believe that the general $\frac{1}{2}$ BPS solutions of $\mathcal{N}=4$ supergravity coupled to n vector multiplets depend on $2+n$ harmonic functions (instead of $4+2n$ harmonic functions for the general $\frac{1}{4}$ BPS solutions).[31] Roughly speaking, the $\frac{1}{4}$ BPS constraints are holomorphic in the complex charges W, Z_{ij} and z^A, whereas the $\frac{1}{2}$ BPS constraints involve a reality condition coming from the complex self-duality of the vector multiplets.

The embedding $\mathrm{SO}(8,2+n) \subset E_{8(8)}$ for $n \le 6$ implies that $\mathcal{N}=4$ supergravity coupled to $n \le 6$ vector multiplets is a consistent truncation of $\mathcal{N}=8$ supergravity. The solutions lying inside $\mathcal{M}_{(p,q)}$ are then $\frac{p+q}{8}$ BPS in $\mathcal{N}=8$ supergravity. Note that the charge matrices which lie in the minimal adjoint orbit of $\mathrm{SO}(6,2+n)$ correspond to $\frac{1}{2}$ BPS solutions in maximal supergravity. This suggests the existence of an intriguing link between minimal adjoint orbits and maximally supersymmetric black holes.

7 Conclusion

In this paper, we have characterised in depth the stationary asymptotically flat solutions of $D=4$ supergravities by a detailed analysis of the duality orbits of the corresponding timelike-reduced Euclidean-signature $D=3$ supergravities. This proceeds initially by analogy with the classification [7] of solutions to three-dimensional supergravities obtained via a spacelike dimensional reduction. A special feature of these Euclidean stationary

[31] The number of harmonic functions is the dimension of the maximal vector space lying inside the relevant stratum (i.e. $\mathbb{R}^{4+2n} \subset \mathcal{M}_{(1,0)}$ and $\mathbb{R}^{2+n} \subset \mathcal{M}_{(2,0)}$ for the $\frac{1}{4}$ and $\frac{1}{2}$ BPS solutions respectively).

solution orbits, however, is the noncompact nature of the isotropy group $\mathfrak{H}^*$, which appears upon making a timelike dimensional reduction to $D = 3$. For $\mathcal{N}$-extended supergravity, the group $\mathfrak{H}^*$ is the product of $\mathrm{Spin}^*(2\mathcal{N})_c \cong \mathrm{Spin}^*(2\mathcal{N})/\ker(S_+)$ (with $\ker(S_+)$ being the kernel of the $\mathrm{Spin}^*(2\mathcal{N})$ chiral Weyl spinor representation) with a group determined by the matter content of the theory.

Rejecting orbits that contain only solutions with naked singularities led us to the quintic characteristic equation (2.21) for the charge matrix $\mathscr{C}$ (2.9). In all but two exceptional cases where the $D = 3$ symmetry groups are $E_{8(8)}$ or $E_{8(-24)}$, this characteristic equation is strengthened to a cubic equation (2.22) for the charge matrix. The $D = 3$ charge matrix $\mathscr{C}$ is the Noether charge for the $D = 3$ duality symmetry; the characteristic equations determine its values in terms of the smaller number of $D = 4$ charges of the same theory (i.e. the gravitational mass and NUT charge and the various electric and magnetic charges of the vector field species). This analysis works for rotating as well as non-rotating solutions; the characteristic equation guarantees that each acceptable orbit passes through some Kerr solution. For pure $\mathcal{N}$-extended supergravity with $\mathcal{N} \leq 5$, the characteristic is equivalent to the Cartan pure spinor condition on the Weyl $\mathrm{Spin}^*(2\mathcal{N})$ spinor $|\mathscr{C}\rangle$.

The characteristic equations involve the BPS parameter $c^2 = \frac{1}{k}\mathrm{Tr}\,\mathscr{C}^2$ (2.20). Extremal rotating solutions have $c^2 = a^2$, where a is the angular momentum parameter. Non-rotating extremal solutions thus have $c^2 = 0$, leading to a key algebraic feature of the extremal solution suborbits: the charge matrix becomes nilpotent — cubic in most cases, quintic in the two E_8 exceptional cases. This allowed us to make contact with extensive studies of nilpotent orbits of noncompact groups in the mathematical literature [40–44].

The extremality condition is not always synonymous with the BPS condition, however. For pure $\mathcal{N} \leq 5$ supergravities, the two conditions are synonymous, but not for $\mathcal{N} = 6$ or $\mathcal{N} = 8$ or any supergravity coupled to vector supermultiplets. Algebraic analysis of the BPS solutions led us to the 'Dirac equation' condition (2.45) in which the charge matrix $\mathscr{C}$ is given an interpretation as a $\mathrm{Spin}^*(2\mathcal{N})$ Weyl spinor, using a creation/annihilation operator construction for the $\mathfrak{so}^*(2\mathcal{N})$ generators. This 'Dirac equation' allows the charge matrix $\mathscr{C}$ to be solved for explicitly in terms of a simple rational function of the $D = 4$ charges.

Having established the relevant families of stationary supergravity solutions, we extended the $D \geq 4$ analysis [15] of active duality transformations (i.e. transformations that leave the asymptotic values of all fields unchanged) to the action of the three-dimensional duality group $\mathfrak{G}$ on these solution families. As in the higher-dimensional cases, in order to preserve the asymptotic values of the fields, the active realisations operate via the quotient of $\mathfrak{G}$ by $\mathfrak{P}_0$, the quotient group of its maximal parabolic subgroup $\mathfrak{P}$ by its defining $\mathbb{R}_+^*$ subgroup. Here, a peculiarity of the non-compact nature of the $D = 3$ scalar isotropy group $\mathfrak{H}^*$ plays a key rôle: although the Iwasawa decomposition remains valid almost everywhere in the moduli space of solutions, it fails precisely on the subspace of extremal solutions. The Iwasawa failure set is not in general homeomorphic to the moduli space of spherically symmetric extremal solutions, however. As a result, there is not in general a well-defined active group action on the whole stationary solution space — some $\mathfrak{G}$ transformations become singular as one approaches the extremal strata. As a result, one has to speak of an 'almost group action' of active transformations on the solution space. We speculate that this curious problem may be resolved in cases where the isotropy group $\mathfrak{H}^*$ is semi-simple, in particular for the $\mathcal{N}=8$ theory.

The results of this $D=3$ duality group analysis should have a bearing on the debate, continuing since the appearance of reference [16], about the extent to which continuous duality symmetries of lower-spacetime-dimensional classical supergravity theories should be replaced by arithmetic subgroups such as $E_8(\mathbb{Z})$ at the quantum level. Although such subgroups certainly exist in the abstract, their concrete realisation as quantum symmetries is problematical because there does not appear to be any way in which the concrete active realisations (cf. (5.54) for an $\mathrm{SL}(2,\mathbb{R})$ example) that we have found for the action of the $D=3$ duality groups $\mathfrak{G}$ on the $D=3$ charges might be consistent with a Dirac quantisation rule.

A Simple duality groups and their five-graded decomposition

Let us review briefly the various simple duality groups in three dimensions which occur in time-like dimensionally reduced four-dimensional theories. We will see that excepted for E_8, all these groups have a five-graded decomposition with respect to which their fundamental representation admits a three-graded decomposition.

Most of these theories can be embedded into supergravity theories. Whenever the symmetric space in which the four-dimensional scalars lie is Kähler, the theory can be embedded into an $\mathcal{N}=1$ supergravity. When the symmetric space is furthermore special Kähler, the theory can moreover be embedded into an $\mathcal{N}=2$ supergravity coupled to several vector multiplets. An $\mathcal{N}=2$ supergravity theory with hypermultiplets always leads to a three-dimensional theory with a reducible symmetric space of scalars, and we do not consider such cases in the present publication. The homogenous special Kähler spaces have been classified in [31]. See [59] for a complete classification.

a) $\mathbf{SL(2+\boldsymbol{n},\mathbb{R})/SO(2,\boldsymbol{n})}$. This coset space corresponds to the dimensional reduction of pure gravity in $4+n$ dimensions. The scalar fields of the four-dimensional theory lie in the coset $GL(n,\mathbb{R})/\mathrm{SO}(n)$ and the five-graded decomposition of $\mathfrak{sl}(2+n,\mathbb{R})$ is as follows

$$\mathfrak{sl}(2+n,\mathbb{R}) \cong \mathbf{1}^{(-2)}\oplus(\square^{(2)}\oplus\overline{\square}^{(-2)})^{(-1)}\oplus\mathbf{1}^{(0)}\oplus\big(\mathfrak{gl}(1,\mathbb{R})\oplus\mathfrak{sl}(n,\mathbb{R})\big)^{(0)}\oplus(\square^{(2)}\oplus\overline{\square}^{(-2)})^{(1)}\oplus\mathbf{1}^{(2)}\,. \tag{A.1}$$

The fundamental representation decomposes as

$$\boldsymbol{n}+\mathbf{2}\cong(\mathbf{1}^{(-1)})^{(-1)}\oplus(\square^{(1)})^{(0)}\oplus(\mathbf{1}^{(-1)})^{(1)}\,. \tag{A.2}$$

b) $\mathbf{SU(1+\boldsymbol{m},1+\boldsymbol{n})/\boldsymbol{S}(U(\boldsymbol{m},1)\times U(1,\boldsymbol{n}))}$. The corresponding four-dimensional theory is the bosonic sector of an $\mathcal{N}=1$ supergravity coupled to $m+n$ abelian vector supermultiplets and mn scalar supermultiplets. In the special case $m=0$, $n=1$, this theory is Maxwell–Einstein theory, which is also the bosonic sector of $\mathcal{N}=2$ pure supergravity. For $m=1$ it is the bosonic sector of an $\mathcal{N}=2$ supergravity coupled to n abelian vector supermultiplets. For $m=3$ it is the bosonic sector of $\mathcal{N}=3$ supergravity theory coupled to n abelian vector supermultiplets. The scalar fields of the four-dimensional theory lie in the

Kähler coset $\mathrm{U}(m,n)/(\mathrm{U}(m)\times\mathrm{U}(n))$ and the five-graded decomposition of $\mathfrak{su}(1+m,1+n)$ is as follows

$$\mathfrak{su}(1+m,1+n)\cong\mathbf{1}^{(-2)}\oplus(\square^{(2)}\oplus\overline{\square}^{(-2)})^{(-1)}\oplus\mathbf{1}^{(0)}\oplus(\mathfrak{u}(1)\oplus\mathfrak{su}(m,n))^{(0)}\oplus(\square^{(2)}\oplus\overline{\square}^{(-2)})^{(1)}\oplus\mathbf{1}^{(2)}\,. \tag{A.3}$$

The complex fundamental representation decomposes as

$$\boldsymbol{m+n+2}\cong(\mathbf{1}_{\mathbb{C}}^{(-1)})^{(-1)}\oplus(\square^{(1)}\oplus\overline{\square}^{(1)})^{(0)}\oplus(\mathbf{1}_{\mathbb{C}}^{(-1)})^{(1)}\,. \tag{A.4}$$

c) $\mathbf{SO(2+m,2+n)/(SO(m,2)\times SO(2,n))}$. For $m=2$ the corresponding four-dimensional theory is the bosonic sector an $\mathcal{N}=2$ supergravity coupled to $1+n$ abelian vector supermultiplets. In the case $m=6$, this is the bosonic sector of $\mathcal{N}=4$ supergravity coupled to n abelian vector supermultiplets. The scalar fields lie in the coset $\mathrm{SO}(2,1)/\mathrm{SO}(2)\times\mathrm{SO}(m,n)/(\mathrm{SO}(m)\times\mathrm{SO}(n))$ and the five-graded decomposition of $\mathfrak{so}(2+m,2+n)$ is as follows

$$\mathfrak{so}(2+m,2+n)\cong\mathbf{1}^{(-2)}\oplus(\square\otimes\square)^{(-1)}\oplus\mathbf{1}^{(0)}\oplus\big(\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{so}(m,n)\big)^{(0)}\oplus(\square\otimes\square)^{(1)}\oplus\mathbf{1}^{(2)}\,. \tag{A.5}$$

It is convenient to consider the irreducible spinor representations $S^{\pm}$ of $\mathrm{Spin}(2+m,2+n)$ and $\mathrm{Spin}(m,n)$, for which we get the decomposition

$$S^{\pm}\cong(\mathbf{1}\otimes S^{\mp})^{(-1)}\oplus(\square\otimes S^{\pm})^{(0)}\oplus(\mathbf{1}\otimes S^{\mp})^{(1)}\,. \tag{A.6}$$

The vector representation decomposes as

$$V\cong(\square\otimes\mathbf{1})^{(-1)}\oplus(\mathbf{1}\otimes\square)^{(0)}\oplus(\square\otimes\mathbf{1})^{(1)}\,. \tag{A.7}$$

d) $\mathbf{SO^*(4+2n)/U(2,n)}$. For $n=0$, $\mathrm{Spin}^*(4)\cong\mathrm{SU}(2)\times\mathrm{SL}(2,\mathbb{R})$ and the corresponding four-dimensional theory is Einstein theory, i.e. the bosonic sector of pure $\mathcal{N}=1$ supergravity. In the case $n=1$, $\mathrm{Spin}^*(6)\cong\mathrm{SU}(1,3)$ and the corresponding four-dimensional theory is the above-discussed bosonic sector of an $\mathcal{N}=1$ supergravity coupled to 2 vector supermultiplets. In general, the corresponding four-dimensional theory is the bosonic sector of an $\mathcal{N}=1$ supergravity coupled to $2n$ abelian vector supermultiplets and $\frac{n(n-1)}{2}$ scalar supermultiplets. The scalar fields of the latter lie in the Kähler coset $\mathrm{SU}(2)/\mathrm{SU}(2)\times\mathrm{SO}^*(2n)/\mathrm{U}(n)$, and the five-graded decomposition of $\mathfrak{so}^*(4+2n)$ is as follows

$$\mathfrak{so}^*(4+2n)\cong\mathbf{1}^{(-2)}\oplus(\square\otimes_{\mathbb{C}}\square)^{(-1)}\oplus\mathbf{1}^{(0)}\oplus\big(\mathfrak{su}(2)\oplus\mathfrak{so}^*(2n)\big)^{(0)}\oplus(\square\otimes_{\mathbb{C}}\square)^{(1)}\oplus\mathbf{1}^{(2)}\,. \tag{A.8}$$

It is convenient to consider the irreducible spinor representations $S^{\pm}$ of $\mathrm{Spin}^*(4+2n)$ and $\mathrm{Spin}^*(2n)$, for which we get the decomposition

$$S^{\pm}\cong(\mathbf{1}\otimes S^{\pm})^{(-1)}\oplus(\square\otimes_{\mathbb{C}}S^{\mp})^{(0)}\oplus(\mathbf{1}\otimes S^{\pm})^{(1)}\,. \tag{A.9}$$

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e) $\mathbf{Sp(2+2}\boldsymbol{n}\mathbf{,\mathbb{R})/U(1,}\boldsymbol{n}\mathbf{)}$. The corresponding four-dimensional theory is the bosonic sector of an $\mathcal{N}=1$ supergravity coupled to n abelian vector supermultiplets and $\frac{n(n+1)}{2}$ scalar supermultiplets. The scalar fields of the latter lie in the Kähler coset $\mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n)$ and the five-graded decomposition of $\mathfrak{sp}(2+2n,\mathbb{R})$ is as follows

$$\mathfrak{sp}(2+2n,\mathbb{R}) \cong \mathbf{1}^{(-2)} \oplus \square^{(-1)} \oplus \mathbf{1}^{(0)} \oplus \mathfrak{sp}(2n,\mathbb{R})^{(0)} \oplus \square^{(1)} \oplus \mathbf{1}^{(2)} \,. \tag{A.10}$$

The fundamental representation decomposes as

$$\mathbf{2}+\mathbf{2}\boldsymbol{n} \cong \mathbf{1}^{(-1)} \oplus \square^{(0)} \oplus \mathbf{1}^{(1)} \,. \tag{A.11}$$

f) $\boldsymbol{G_{2(2)}}\mathbf{/(SU(1,1)\times SU(1,1))}$. The corresponding four-dimensional theory is the bosonic sector of an $\mathcal{N}=2$ supergravity theory coupled to one vector supermultiplet, which corresponds itself to the dimensional reduction of minimal supergravity in five dimensions. The scalar fields of the four-dimensional theory lie in the special Kähler coset $\mathrm{SU}(1,1)/\mathrm{U}(1)$ and the five-graded decomposition of $\mathfrak{g}_{2(2)}$ is as follows

$$\mathfrak{g}_{2(2)} \cong \mathbf{1}^{(-2)} \oplus \square\!\square\!\square^{(-1)} \oplus \mathbf{1}^{(0)} \oplus \mathfrak{su}(1,1)^{(0)} \oplus \square\!\square\!\square^{(1)} \oplus \mathbf{1}^{(2)} \,. \tag{A.12}$$

The fundamental representation decomposes as

$$\mathbf{7} \cong \square^{(-1)} \oplus \square\!\square^{(0)} \oplus \square^{(1)} \,. \tag{A.13}$$

g) $\boldsymbol{F_{4(4)}}\mathbf{/(SU(1,1)\times Sp(6,\mathbb{R}))}$. The corresponding four-dimensional theory is the bosonic sector of the real magic $\mathcal{N}=2$ supergravity, which admits 6 abelian vector supermultiplets. The scalar fields of the latter lie in the special Kähler coset $\mathrm{Sp}(6,\mathbb{R})/\mathrm{U}(3)$ and the five-graded decomposition of $\mathfrak{f}_{4(4)}$ is as follows

$$\mathfrak{f}_{4(4)} \cong \mathbf{1}^{(-2)} \oplus \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}^{(-1)} \oplus \mathbf{1}^{(0)} \oplus \mathfrak{sp}(6,\mathbb{R})^{(0)} \oplus \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}^{(1)} \oplus \mathbf{1}^{(2)} \,. \tag{A.14}$$

The fundamental representation decomposes as

$$\mathbf{26} \cong \square^{(-1)} \oplus \begin{smallmatrix}\square\\\square\end{smallmatrix}^{(0)} \oplus \square^{(1)} \,. \tag{A.15}$$

h) $\boldsymbol{E_{6(6)}}\mathbf{/Sp(8,\mathbb{R})}$. The scalar fields of the corresponding four-dimensional theory lie in the coset $\mathrm{SL}(6,\mathbb{R})/\mathrm{SO}(6)$ and the five-graded decomposition of $\mathfrak{e}_{6(6)}$ is as follows

$$\mathfrak{e}_{6(6)} \cong \mathbf{1}^{(-2)} \oplus \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}^{(-1)} \oplus \mathbf{1}^{(0)} \oplus \mathfrak{sl}(6,\mathbb{R})^{(0)} \oplus \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}^{(1)} \oplus \mathbf{1}^{(2)} \,. \tag{A.16}$$

The fundamental representation decomposes as

$$\mathbf{27} \cong \overline{\square}^{(-1)} \oplus \begin{smallmatrix}\square\\\square\end{smallmatrix}^{(0)} \oplus \overline{\square}^{(1)} \,. \tag{A.17}$$

i) $E_{6(2)}/(\mathrm{SU}(1,1) \times \mathrm{SU}(3,3))$. The corresponding four-dimensional theory is the bosonic sector of the complex magic $\mathcal{N} = 2$ supergravity, which admits 9 abelian vector supermultiplets. The scalar fields of the latter lie in the special Kähler coset $\mathrm{SU}(3,3)/S(\mathrm{U}(3) \times \mathrm{U}(3))$ and the five-graded decomposition of $\mathfrak{e}_{6(2)}$ is as follows

$$\mathfrak{e}_{6(2)} \cong \mathbf{1}^{(-2)} \oplus \ydiagram{1,1,1}_{+}^{(-1)} \oplus \mathbf{1}^{(0)} \oplus \mathfrak{su}(3,3)^{(0)} \oplus \ydiagram{1,1,1}_{+}^{(1)} \oplus \mathbf{1}^{(2)}\,, \tag{A.18}$$

where the $_+$ subscript states for complex-self-duality. The complex fundamental representation decomposes as

$$\mathbf{27} \cong \overline{\square}^{(-1)} \oplus \ydiagram{1,1}^{(0)} \oplus \overline{\square}^{(1)}\,. \tag{A.19}$$

j) $E_{6(-14)}/(\mathrm{U}(1)\times \mathrm{SO}^*(10))$. The corresponding four-dimensional theory is the bosonic sector of $\mathcal{N} = 5$ supergravity. The scalar fields lie in the coset $\mathrm{SU}(5,1)/\mathrm{U}(5)$ and the five-graded decomposition of $\mathfrak{e}_{6(-14)}$ is as follows

$$\mathfrak{e}_{6(-14)} \cong \mathbf{1}^{(-2)} \oplus \ydiagram{1,1,1}_{+}^{(-1)} \oplus \mathbf{1}^{(0)} \oplus \mathfrak{su}(5,1)^{(0)} \oplus \ydiagram{1,1,1}_{+}^{(1)} \oplus \mathbf{1}^{(2)}\,. \tag{A.20}$$

The complex fundamental representation decomposes as

$$\mathbf{27} \cong \overline{\square}^{(-1)} \oplus \ydiagram{1,1}^{(0)} \oplus \overline{\square}^{(1)}\,. \tag{A.21}$$

k) $E_{7(7)}/\mathrm{SU}(4,4)$. The scalar fields of the corresponding four-dimensional theory lie in the coset $\mathrm{SO}(6,6)/(\mathrm{SO}(6) \times \mathrm{SO}(6))$ and the five-graded decomposition of $\mathfrak{e}_{7(7)}$ is as follows

$$\mathfrak{e}_{7(7)} \cong \mathbf{1}^{(-2)} \oplus S_+^{(-1)} \oplus \mathbf{1}^{(0)} \oplus \mathfrak{spin}(6,6)^{(0)} \oplus S_+^{(1)} \oplus \mathbf{1}^{(2)} \tag{A.22}$$

The fundamental representation decomposes as

$$\mathbf{56} \cong V^{(-1)} \oplus S_-^{(0)} \oplus V^{(1)}\,, \tag{A.23}$$

where $S_\pm$ are the 32-dimensional Majorana–Weyl representations of $\mathrm{Spin}(6,6)$ and V is the vector representation of $\mathrm{SO}(6,6)$.

l) $E_{7(-5)}/(\mathrm{SU}(1,1) \times \mathrm{SO}^*(12))$. The corresponding four-dimensional theory is the bosonic sector of both $\mathcal{N} = 6$ supergravity and of the quaternionic magic $\mathcal{N} = 2$ supergravity, which admits 15 abelian vector supermultiplets. The scalar fields lie in the special Kähler coset $\mathrm{SO}^*(12)/\mathrm{U}(6)$ and the five-graded decomposition of $\mathfrak{e}_{7(-5)}$ is as follows

$$\mathfrak{e}_{7(-5)} \cong \mathbf{1}^{(-2)} \oplus S_+^{(-1)} \oplus \mathbf{1}^{(0)} \oplus \mathfrak{spin}^*(12)^{(0)} \oplus S_+^{(1)} \oplus \mathbf{1}^{(2)}\,. \tag{A.24}$$

The complex fundamental representation decomposes as

$$\mathbf{56} \cong V^{(-1)} \oplus S_-^{(0)} \oplus V^{(1)}\,, \tag{A.25}$$

where S_+ is the Majorana–Weyl representation of $\mathrm{Spin}^*(12)$, whereas V and S_- are complex, respectively vector and Weyl spinor, representations of $\mathrm{Spin}^*(12)$.

m) $E_{7(-25)}/(\mathrm{SO}(2)\times E_{6(-14)})$. The corresponding four-dimensional theory is an $\mathcal{N}=1$ supergravity coupled to 16 abelian vector supermultiplets and 10 scalar supermultiplets. The scalar fields of the latter lie in the Kähler coset $\mathrm{SO}(2,10)/(\mathrm{SO}(2)\times\mathrm{SO}(10))$ and the five-graded decomposition of $\mathfrak{e}_{7(-25)}$ is as follows

$$\mathfrak{e}_{7(-25)} \cong \mathbf{1}^{(-2)} \oplus S_+^{(-1)} \oplus \mathbf{1}^{(0)} \oplus \mathfrak{spin}(2,10)^{(0)} \oplus S_+^{(1)} \oplus \mathbf{1}^{(2)} . \tag{A.26}$$

The fundamental representation decomposes as

$$\mathbf{56} \cong V^{(-1)} \oplus S_-^{(0)} \oplus V^{(1)} , \tag{A.27}$$

where $S_\pm$ are the 32-dimensional Majorana–Weyl representations of $\mathrm{Spin}(2,10)$ and V is the vector representation of $\mathrm{SO}(2,10)$.

n) $E_{8(8)}/\mathrm{SO}^*(16)$. The corresponding four-dimensional theory is the bosonic sector of $\mathcal{N}=8$ supergravity. The scalar fields of the latter lie in the coset $E_{7(7)}/(\mathrm{SU}(8)/\mathbb{Z}_2)$ and the five-graded decomposition of $\mathfrak{e}_{8(8)}$ is as follows

$$\mathfrak{e}_{8(8)} \cong \mathbf{1}^{(-2)} \oplus \mathbf{56}^{(-1)} \oplus \mathbf{1}^{(0)} \oplus \mathfrak{e}_{7(7)}^{(0)} \oplus \mathbf{56}^{(1)} \oplus \mathbf{1}^{(2)} . \tag{A.28}$$

The fundamental is the adjoint, and the **3875** representation is also five-graded,

$$\mathbf{3875} \cong \mathbf{133}^{(-2)} \oplus \mathbf{56}^{(-1)} \oplus \mathbf{912}^{(-1)} \oplus \mathbf{1}^{(0)} \oplus \mathbf{133}^{(0)} \oplus \mathbf{1539}^{(0)} \oplus \mathbf{56}^{(1)} \oplus \mathbf{912}^{(1)} \oplus \mathbf{133}^{(2)} . \tag{A.29}$$

o) $E_{8(-24)}/(\mathrm{SU}(1,1)\times E_{7(-25)})$. The corresponding four-dimensional theory is the bosonic sector of the octonionic magic $\mathcal{N}=2$ supergravity, which admits 27 abelian vector supermultiplets. The scalar fields of the latter lie in the special Kähler coset $E_{7(-25)}/(\mathrm{U}(1)\times E_{6(-78)})$ and the five-graded decomposition of $\mathfrak{e}_{8(-24)}$ is as follows

$$\mathfrak{e}_{8(-24)} \cong \mathbf{1}^{(-2)} \oplus \mathbf{56}^{(-1)} \oplus \mathbf{1}^{(0)} \oplus \mathfrak{e}_{7(-25)}^{(0)} \oplus \mathbf{56}^{(1)} \oplus \mathbf{1}^{(2)} . \tag{A.30}$$

The fundamental is the adjoint, and the **3875** representation is also five-graded,

$$\mathbf{3875} \cong \mathbf{133}^{(-2)} \oplus \mathbf{56}^{(-1)} \oplus \mathbf{912}^{(-1)} \oplus \mathbf{1}^{(0)} \oplus \mathbf{133}^{(0)} \oplus \mathbf{1539}^{(0)} \oplus \mathbf{56}^{(1)} \oplus \mathbf{912}^{(1)} \oplus \mathbf{133}^{(2)} . \tag{A.31}$$

B Spin$^*(2\mathcal{N})$ and its representations

In this appendix we summarise some pertinent results concerning the group $\mathrm{Spin}^*(2\mathcal{N})$ and its spinorial representations, comparing them to the corresponding representations of the compact group $\mathrm{Spin}(2\mathcal{N})$. We also refer to ref. [60] for a detailed discussion of the algebra $\mathfrak{so}^*(2\mathcal{N})$. These two groups are different real forms of the same complex Lie group $\mathrm{Spin}(2\mathcal{N},\mathbb{C})$, with the compact $\mathrm{U}(\mathcal{N})$ group as their intersection. Because their complex representations are thus the same, it will be convenient to analyse these representations in the basis $\oplus_n \bigwedge^n \mathbb{C}^{\mathcal{N}}$. For this purpose, we will make use of the fermionic creation and annihilation operators a_i and $a^i \equiv (a_i)^\dagger$ already introduced in section 2.2 (with $i,j,\dots \in \{1,\dots\mathcal{N}\}$)

$$\{a_i,a_j\} = \{a^i,a^j\} = 0 \quad , \qquad \{a_i,a^j\} = \delta_i^j . \tag{B.1}$$

Since all the generators of both $\mathrm{Spin}^*(2\mathcal{N})$ and $\mathrm{Spin}(2\mathcal{N})$ commute with the diagonal matrix $(-1)^n$, the spinor representations decompose into chiral and anti-chiral Weyl spinor representations $\oplus_p \bigwedge^{2p}\mathbb{C}^{\mathcal{N}}$ and $\oplus_p \bigwedge^{2p+1}\mathbb{C}^{\mathcal{N}}$, respectively. These representations can thus be obtained by acting with an even or an odd number of creation operators on the vacuum $|0\rangle$, that is, we have

$$|\mathscr{C}\rangle = \Big(W + Z_{ij}a^i a^j + \Sigma_{ijkl}a^i a^j a^k a^l + \dots\Big)|0\rangle \tag{B.2}$$

for the chiral and

$$|\mathscr{C}\rangle = \Big(\psi_i a^i + \chi_{ijk}a^i a^j a^k + \dots\Big)|0\rangle \tag{B.3}$$

for the antichiral representations, respectively.

The groups $\mathrm{Spin}(2\mathcal{N})$ and $\mathrm{Spin}^*(2\mathcal{N})$ are respectively the two real forms of $\mathrm{Spin}(2\mathcal{N},\mathbb{C})$ defined by the conditions

$$U^\dagger = U^{-1} \quad [\text{for } \mathrm{Spin}(2\mathcal{N})] \qquad \text{and} \qquad U^\dagger = \beta U^{-1}\beta \quad [\text{for } \mathrm{Spin}^*(2\mathcal{N})] \tag{B.4}$$

where the matrix β is defined to act on both $\oplus_p \bigwedge^{2p}\mathbb{C}^{\mathcal{N}}$ and $\oplus_p \bigwedge^{2p+1}\mathbb{C}^{\mathcal{N}}$ as $(-1)^p$. The generators of the $\mathfrak{u}(\mathcal{N})$ maximal subalgebra of both algebras are defined in terms of the anti-Hermitean parameters $\Lambda_i{}^j = -\Lambda^j{}_i$ as

$$K(\Lambda) = \frac{1}{2}\Lambda_i{}^j\,[a^i, a_j] \quad \Rightarrow \qquad K(\Lambda)^\dagger = -K(\Lambda) \tag{B.5}$$

The remaining generators depend on the antisymmetric tensors Λ_{ij} of $\mathrm{U}(\mathcal{N})$: for $\mathrm{Spin}(2\mathcal{N})$ we have the anti-Hermitean generators

$$T(\Lambda) = \Lambda_{ij}a^i a^j + \Lambda^{ij}a_i a_j \quad \Rightarrow \qquad T(\Lambda)^\dagger = -T(\Lambda)\,, \tag{B.6}$$

whereas for the non-compact real form $\mathrm{Spin}^*(2\mathcal{N})$ we have

$$T^*(\Lambda) = \Lambda_{ij}a^i a^j - \Lambda^{ij}a_i a_j \quad \Rightarrow \qquad T^*(\Lambda)^\dagger = T^*(\Lambda)\ . \tag{B.7}$$

With the above definition of β it follows that

$$\beta G(\Lambda)\beta = -G(\Lambda)^\dagger \tag{B.8}$$

for both $G = K$ and $G = T^*$. From these formulas we see that the conjugate of a spinor $|\lambda\rangle$ must be defined as

$$\langle\lambda| \equiv \big(|\lambda\rangle\big)^\dagger \quad [\text{for } \mathrm{Spin}(2\mathcal{N})] \qquad \text{and} \qquad {}^*\langle\lambda| \equiv \big(|\lambda\rangle\big)^\dagger\beta \quad [\text{for } \mathrm{Spin}^*(2\mathcal{N})]\ . \tag{B.9}$$

Let us also record the expression for the $\mathfrak{u}(1)$ generator of $\mathfrak{u}(\mathcal{N})$ in terms of oscillators, viz.

$$J \equiv \frac{1}{2}\,[a^i, a_i] = a^i a_i - \frac{1}{2}\mathcal{N}\,, \tag{B.10}$$

which permits one to re-express β as

$$\beta|_{\oplus_p \bigwedge^{2p}\mathbb{C}^{\mathcal{N}}} \equiv (-1)^{\frac{J}{2}+\frac{\mathcal{N}}{4}} \qquad \beta|_{\oplus_p \bigwedge^{2p-1}\mathbb{C}^{\mathcal{N}}} \equiv (-1)^{\frac{J}{2}+\frac{\mathcal{N}}{4}+\frac{1}{2}} \tag{B.11}$$

or the chiral and the antichiral Weyl spinors respectively.

As for Spin($2\mathcal{N}$), the centre of Spin$^*(2\mathcal{N})$ is generated by the group elements $e^{i\pi J}$ and $-\mathbb{1}$. For odd $\mathcal{N}$, we have $\left(e^{i\pi J}\right)^2 = -\mathbb{1}$ and the centre is $\mathbb{Z}_4$. For even $\mathcal{N}$, $\left(e^{i\pi J}\right)^2 = \mathbb{1}$ and the centre is $\mathbb{Z}_2 \times \mathbb{Z}_2$. In the latter case, the $\mathbb{Z}_2$ subgroup generated by the group element $e^{i\pi\left(J+\frac{\mathcal{N}}{2}\right)}$ acts trivially on the chiral Weyl spinor representation, whereas it acts as $-\mathbb{1}$ on the anti-chiral Weyl spinor representation and the vector representation. The chiral Weyl spinor representation is thus a representation of the group Spin$^*(2\mathcal{N})/\mathbb{Z}_2$, and it is this latter which appears in the definition of the scalar-field coset space, i.e.

$$\mathrm{U}(1) \times_{\mathbb{Z}_2} \frac{\mathrm{Spin}^*(8)}{\mathbb{Z}_2} \cong \mathrm{SO}(2) \times_{\mathbb{Z}_2} \mathrm{SO}(2,6), \qquad \mathrm{SU}(1,1) \times_{\mathbb{Z}_2} \frac{\mathrm{Spin}^*(12)}{\mathbb{Z}_2}, \qquad \frac{\mathrm{Spin}^*(16)}{\mathbb{Z}_2}$$

for $\mathcal{N} = 4$, 6 and 8 respectively.

The (anti-)chiral representations given above are not always irreducible. To analyse the values of $\mathcal{N}$ for which this happens, we first note that one can define certain anti-involutions or pseudo-anti-involutions for both Spin($2\mathcal{N}$) and Spin$^*(2\mathcal{N})$ by making use of the SU($\mathcal{N}$) — preserving Hodge star operator $\star$ which maps $\oplus_n \bigwedge^n \mathbb{C}^{\mathcal{N}}$ to its conjugate. The Hodge star obeys

$$\star^2 = (-1)^{n(\mathcal{N}-n)} \; . \tag{B.12}$$

The definition of the respective (pseudo-)anti-involutions, which we denote here by $\ast$ and $\ast^*$, respectively, involves extra sign factors, as we will explain below. Let us now analyse the different cases in turn.

For $\mathcal{N}$ odd there is no difference between the spinor representations of Spin($2\mathcal{N}$) and Spin$^*(2\mathcal{N})$. In this case, the (pseudo)-anti-involution does not commute with $(-1)^n$ and therefore the spinor and its conjugate are simply the two inequivalent irreducible complex spinor representations, for both Spin($2\mathcal{N}$) and Spin$^*(2\mathcal{N})$. For $\mathcal{N}$ even, on the other hand, both $\ast$ and $\ast^*$ commute with $(-1)^n$ and the Weyl spinor representations become reducible if $\ast$ and $\ast^*$ are anti-involutions, that is, if they square to one on these subspaces.

For the reader's convenience, we first recall some familiar results for the compact real form Spin($2\mathcal{N}$) (cf. [32]). For Spin($8M$), the operation $\ast$ is defined on $\oplus_p \bigwedge^{2p} \mathbb{C}^{4M}$ as

$$\ast \oplus_{p=1}^{2M} \psi_{(2p)} \equiv \oplus_{p=1}^{2M} \left((-1)^p \star \overline{\psi_{(4M-2p)}} \right) \; . \tag{B.13}$$

Since $\star^2 = 1$ on even forms, $\ast^2 = 1$ in this case. On the anti-chiral spinor $\oplus_p \bigwedge^{2p-1} \mathbb{C}^{4M}$, the formula is

$$\ast \oplus_{p=1}^{2M} \psi_{(2p-1)} \equiv \oplus_{p=1}^{2M} \left((-1)^p \star \overline{\psi_{(4M-2p+1)}} \right) \; . \tag{B.14}$$

Now, $\star^2 = -1$ on odd forms in even dimensions, but $(-1)^p(-1)^{2M-p+1} = -1$ so that $\ast^2 = 1$. Therefore, in both cases, one can impose the reality condition $\ast|\lambda\rangle = |\lambda\rangle$, thereby reducing the Weyl spinors to Majorana-Weyl spinors.

For Spin($8M+4$), $\ast$ is defined to act on $\oplus_p \bigwedge^{2p} \mathbb{C}^{4M+2}$ as

$$\ast \oplus_{p=1}^{2M+1} \psi_{(2p)} \equiv \oplus_{p=1}^{2M} \left((-1)^p \star \overline{\psi_{(4M+2-2p)}} \right) \; . \tag{B.15}$$

	vector	chiral spinor	antichiral spinor	centre
Spin$(8M)$	real	real	real	$\mathbb{Z}_2 \times \mathbb{Z}_2$
Spin$(8M+4)$	real	pseudo-real	pseudo-real	$\mathbb{Z}_2 \times \mathbb{Z}_2$
Spin$(4M+2)$	real	complex	complex	$\mathbb{Z}_4$

Table 10. Reality conditions of Spin$(2\mathcal{N})$ representations.

Although $\star^2 = 1$ on even forms, $(-1)^p(-1)^{2M+1-p} = -1$ and $\star^2 = -1$ in this case. Similarly, on $\oplus_p \bigwedge^{2p-1} \mathbb{C}^{4M+2}$, one has

$$\star \oplus_{p=1}^{2M+1} \psi_{(2p-1)} \equiv \oplus_{p=1}^{2M} \Big((-1)^p \star \overline{\psi_{(4M-2p-1)}} \Big) . \tag{B.16}$$

Because $\star^2 = -1$ on odd forms in even dimensions and $(-1)^p(-1)^{2M-p} = 1$ one obtains again $\star^2 = -1$. Consequently, the Spin$(8M+4)$ Weyl spinor representations are irreducible, though pseudo-real. Altogether we have thus rederived the well-known result

$$\star\star = (-1)^{\frac{\mathcal{N}}{2}} \tag{B.17}$$

on $\oplus_n \bigwedge^n \mathbb{C}^{\mathcal{N}}$ for even $\mathcal{N}$.

For the non-compact real form Spin$^*(4M)$ and its Weyl representations, the operation $\star^* \equiv \beta\star$ is defined on $\oplus_p \bigwedge^{2p} \mathbb{C}^{2M}$ as

$$\star^* \oplus_{p=1}^{M} \psi_{(2p)} \equiv \oplus_{p=1}^{M} \Big(\star \overline{\psi_{(2M-2p)}} \Big) . \tag{B.18}$$

Because $\star^2 = 1$ on even forms, one gets $\star^*\star^* = 1$ in this case. Similarly, on $\oplus_p \bigwedge^{2p-1} \mathbb{C}^{2M}$ one has

$$\star^* \oplus_{p=1}^{M} \psi_{(2p-1)} \equiv \oplus_{p=1}^{2M} \Big(\star \overline{\psi_{(2M-2p+1)}} \Big) . \tag{B.19}$$

Now $\star^2 = -1$ on odd forms in even dimensions whence $\star^*\star^* = -1$ in this case. We thus conclude that the chiral Weyl spinor representation of Spin$^*(4M)$ always decomposes into two equivalent Majorana–Weyl representations, whereas the anti-chiral Weyl spinor representations of Spin$^*(4M)$ are always pseudo-real, hence irreducible. We have thus shown that the analogue of (B.17) reads, for $\mathcal{N} = 4M$,

$$\star^*\star^* = \begin{cases} +1 & \text{for chiral spinors} \\ -1 & \text{for anti-chiral spinors} . \end{cases}$$

These properties are summarised in table 10 and 11, for Spin$(2\mathcal{N})$ and Spin$^*(2\mathcal{N})$, respectively.[32]

When $\mathcal{N} = 4M$, the above results for Spin$^*(2\mathcal{N})$ would seem to pose a problem for the boson-fermion balance required by supersymmetry, because unlike for Spin$(2\mathcal{N})$ where both

[32]For $\mathcal{N} = 4$ (i.e. $M = 2$), Spin$^*(8) \cong$ Spin$(2,6)$ and, owing to triality, the complex vector representation of SO$^*(8)$ is isomorphic to the antichiral Weyl spinor representation of Spin$(2,6)$, which leads to the existence of a sixteen real dimensional Spin$(2,6)$ SU(2) — Majorana representation.

	vector	chiral spinor	antichiral spinor	centre
$\mathrm{Spin}^*(4M)$	pseudo-real	real	pseudo-real	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathrm{Spin}^*(4M+2)$	pseudo-real	complex	complex	$\mathbb{Z}_4$

Table 11. Reality conditions of $\mathrm{Spin}^*(2\mathcal{N})$ representations.

chiral and antichiral spinors share the same number of degrees of freedom, the antichiral representation requires twice as many degrees of freedom as the chiral one. Fortunately, at this point the presence of the spatial rotation group SU(2) comes to our rescue: namely, the spinor fields transform not only under $\mathrm{Spin}^*(2\mathcal{N})$ but under $\mathrm{SU}(2)\times\mathrm{Spin}^*(2\mathcal{N})$ for any $\mathcal{N}$. The existence of the SU(2) invariant tensor $\varepsilon_{\alpha\beta}$ allows us to impose the representation halving condition

$$(\star^*|\lambda\rangle)^\alpha = \varepsilon^{\alpha\beta}|\lambda\rangle_\beta \tag{B.20}$$

replacing the Majorana-Weyl condition (which would not work by itself) by a *symplectic Majorana-Weyl* condition. In this way the boson-fermion balance necessary for supersymmetry can be restored.

Let us explain a bit more explicitly how this works for $\mathcal{N} = 6$ and $\mathcal{N} = 8$. For simplicity of notation, we will now write $\star^* \equiv \star$ and give all formulas with two signs, the upper ones corresponding to the non-compact group $\mathrm{Spin}^*(2\mathcal{N})$, and the lower ones to the compact group $\mathrm{Spin}(2\mathcal{N})$. For $\mathcal{N} = 6$, the chiral spinor can be written as

$$|\mathscr{C}\rangle = \Big(W + Z_{ij}a^ia^j + \frac{1}{4!}\varepsilon_{ijklmn}\Sigma^{ij}a^ka^la^ma^n + \frac{1}{6!}\varepsilon_{ijklmn}Za^ia^ja^ka^la^ma^n\Big)|0\rangle \tag{B.21}$$

on which the coset generators act as

$$\delta|\mathscr{C}\rangle = \Big(\Lambda_{ij}a^ia^j \mp \Lambda^{ij}a_ia_j\Big)|\mathscr{C}\rangle \tag{B.22}$$

The (pseudo-)anti-involution is defined as follows

$$\star|\mathscr{C}\rangle := \Big(\bar{Z} \pm \Sigma_{ij}a^ia^j + \frac{1}{4!}\varepsilon_{ijklmn}Z^{ij}a^ka^la^ma^n \pm \frac{1}{6!}\varepsilon_{ijklmn}\bar{W}a^ia^ja^ka^la^ma^n\Big)|0\rangle \tag{B.23}$$

and is preserved by the transformations

$$\begin{aligned} \delta W &= 2\Lambda^{ij}Z_{ij} \qquad & \delta Z_{ij} &= \Lambda_{ij}W + \frac{1}{2}\varepsilon_{ijklmn}\Lambda^{kl}\Sigma^{mn} \\ \delta Z &= 2\Lambda_{ij}\Sigma^{ij} \qquad & \delta\Sigma^{ij} &= \Lambda^{ij}Z + \frac{1}{2}\varepsilon^{ijklmn}\Lambda_{kl}Z_{mn}\ . \end{aligned} \tag{B.24}$$

For an antichiral spinor we have

$$|\chi\rangle \equiv \Big(\psi_ia^i + \chi_{ijk}a^ia^ja^k + \frac{1}{5!}\varepsilon_{ijklmn}\chi^na^ia^ja^ka^la^m\Big)\,|0\rangle \tag{B.25}$$

and the (pseudo-)anti-involution reads

$$\star\,|\chi\rangle \equiv \Big(\mp\chi_ia^i + \frac{1}{6!}\varepsilon_{ijklmn}\,\chi^{lmn}\,a^ia^ja^k \pm \frac{1}{5!}\varepsilon_{ijklmn}\,\psi^n\,a^ia^ja^ka^la^m\Big)\,|0\rangle\ . \tag{B.26}$$

Finally, for maximal supergravity, the relevant group is Spin*(16), and a chiral Weyl spinor can be represented by the state

$$|\mathscr{C}\rangle \equiv \Big(W + Z_{ij}a^i a^j + \Sigma_{ijkl}a^i a^j a^k a^l + Z_{ijklmn}a^i a^j a^k a^l a^m a^n \\ +W_{ijklmnpq}a^i a^j a^k a^l a^m a^n a^p a^q\Big)\,|0\rangle \ . \qquad (B.27)$$

The anti-involution is then (where the lower sign is for Spin(16))

$$\star\,|\mathscr{C}\rangle \equiv \Big(\varepsilon_{ijklmnpq}\,W^{ijklmnpq} \pm \frac{1}{2}\varepsilon_{ijklmnpq}\,Z^{klmnpq}\,a^i a^j \\ + \frac{1}{4!}\varepsilon_{ijklmnpq}\,\Sigma^{mnpq}\,a^i a^j a^k a^l \pm \frac{1}{6!}\varepsilon_{ijklmnpq}\,Z^{pq}\,a^k a^l a^m a^n a^p a^q \\ + \frac{1}{8!}\varepsilon_{ijklmnpq}\,\bar{W}\,a^i a^j a^k a^l a^m a^n a^p a^q\Big)\,|0\rangle \ . \qquad (B.28)$$

Similarly, for an antichiral spinor one has

$$|\chi\rangle \equiv \Big(\psi_i a^i + \chi_{ijk}a^i a^j a^k + \chi_{ijklm}a^i a^j a^k a^l a^m + \psi_{ijklmnp}a^i a^j a^k a^l a^m a^n a^p\Big)\,|0\rangle \ . \qquad (B.29)$$

The anti-involution $\star$ of Spin(16) corresponds to the pseudo-anti-involution of Spin*(16),

$$\star\,|\chi\rangle \equiv \Big(\pm\varepsilon_{ijklmnpq}\,\psi^{jklmnpq}\,a^i + \frac{1}{3!}\varepsilon_{ijklmnpq}\,\chi^{lmnpq}\,a^i a^j a^k \\ \pm\frac{1}{5!}\varepsilon_{ijklmnpq}\,\chi^{npq}\,a^i a^j a^k a^l a^m + \frac{1}{7!}\varepsilon_{ijklmnpq}\,\psi^{q}\,a^i a^j a^k a^l a^m a^n a^p\Big)\,|0\rangle \ . \qquad (B.30)$$

Acknowledgments

We would like thank Ling Bao, P. Breitenlohner, D. Ž. Đoković, A. Kleinschmidt, I. Melnikov, B. Pioline, D. Vogan and B. de Wit for discussions related to this work. The research of K.S.S. was supported in part by the EU under contract MRTN-CT-2004-005104, by the STFC under rolling grant PP/D0744X/1 and by the Alexander von Humboldt Foundation through the award of a Research Prize. K.S.S. would like to thank the Albert Einstein Institute and CERN for hospitality during the course of the work.

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